R-composition of Lyapunov functions

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Abstract—This paper introduces the use of R-functions to compose (R-composition) basic simple Lyapunov functions, like the conventional quadratic ones, to obtain a larger variety of functions. R-functions represent the natural extension of Boolean operators to real-valued functions and provide the basic tools to compute the analytic expression of intersection and union operations in a geometric setting. In the framework of Lyapunov approaches to prove stability of a dynamical system, the union of Lyapunov functions computed through the R-function approach is still a Lyapunov function. Moreover, as each Lyapunov function defines the shape and the orientation of a correspondent geometric Largest Estimate of the Domain of Attraction (LEDA), then the LEDA associated to the union of several Lyapunov functions corresponds to the union of the single LEDAs. R-composition of Lyapunov functions thus corresponds to a non-conventional Lyapunov function which can be used to improve the estimate of the Region of Asymptotic Stability (RAS) and at the same time to introduce more freedom in the choice of the shape of the correspondent level sets, that in general are non-convex. An example of the R-composition of Lyapunov functions is illustrated to solve a classic RAS estimation problem.

I. INTRODUCTION

One of the most challenging differences between linear and nonlinear systems is that in the latter case there may be many equilibrium points, each one of them eventually with a very irregular basin of attraction. The conventional approach to prove the stability of an equilibrium, also known as first method of Lyapunov, consists in linearising the system with respect to the equilibrium of interest and checking the sign of the eigenvalues of the resulting transition matrix. Major difficulties arise whenever the previous approach is not conclusive (e.g. for continuous dynamical systems where no eigenvalue is positive and at least one of them is zero), or if one is interested in investigating the size of the Region of Asymptotic Stability (RAS). The difficulty arises because no systematic approaches exist to tackle such a problem, and each of the methods known from literature could be more or less effective depending on the equations describing the dynamical systems. The most popular approach to prove stability is the second method of Lyapunov and requires building a suitable function of the state of the dynamical system. Several empirical rules are known to build an effective Lyapunov Function (LF); if the dynamical system is a physical one, a classical approach chooses its energy as a candidate LF. If such an assumption makes sense, stable systems lose energy over time and eventually reach some final resting state. Not only can this approach be conclusive about stability in case the first method fails, but it also provides a first approximation of the RAS. Therefore it is obvious that most of the efforts of using a Lyapunov approach consist in tailoring a function that can both be conclusive about stability and at the same time be as much accurate to estimate the RAS as possible. In this view, the best choice of Lyapunov function depends on the RAS of a particular dynamical system, while a systematic choice within one family of LFs, as for instance the conventional quadratic functions (i.e. of the kind of $x^TPx$, where $P$ is a positive definite matrix), constrains the search of RAS within ellipsoidal sets and in many situations it may provide a poor approximation.

A first approach and a survey on the techniques to the estimation of RAS can be found in [1] and [2], where Zubov, La Salle and non-Lyapunov methods are described. More recently there is a strong trend in estimating the RAS by composing several LFs. This has led for instance to the use of piecewise-linear LFs [3]-[4], piecewise affine Lyapunov functions [5], lifting technique [6] and composite quadratic Lyapunov functions [7] and [8]. These approaches are mainly motivated by the fact that an a priori choice of the Lyapunov function defines the shape and orientation of the Largest Estimate of the Domain of Attraction (LEDA). On the other hand, the composition of several Lyapunov functions provides more freedom on the shape of the level sets, and to refine the estimation of the effective RAS. A recent general approach that is based on the union of continuous families of Lyapunov estimates is [9], while in [10] the composition of LFs is motivated by the decomposition of a dynamical system into simpler one-dimensional interconnected systems.

Analytic procedures to enlarge RAS estimations have been proposed in the literature to tackle special families of dynamical systems, and in particular for linear systems subject to
actuator saturation [11] and [12]. The enlargement of the RAS estimation requires the use of optimisation algorithms and the most popular way is to exploit LMI methods [13].

In this paper a systematic way of composing Lyapunov functions through the use of R-functions is proposed, thus leading to what in the remainder of the paper will be called R-composition, to obtain richer and more flexible LFs. The theory of R-functions provides the equivalent of logic (Boolean) functions (e.g. AND, OR, NOT) to compute geometrical operations like intersections, unions and complementary sets. Although R-functions were first introduced by Rvachev about forty years ago [14]-[15], they have never been used for control problems, at least to the authors' knowledge, although they provide several benefits when used for describing geometrical regions. For this reason, the main benefit of the composed LF is more evident if one looks at its geometrical counterpart, since the LEDA associated to the composed LF corresponds to the union of the LEDAs associated to each LF alone. This property is potentially very powerful because it is usually very convenient to test simple functions, like the conventional quadratic ones, as candidate LFs. On the other hand, it is more challenging to guess a priori a strange and irregular LF that would provide a LEDA that is a more accurate estimate of the unknown RAS. Nevertheless, we suggest here that first several different quadratic functions should be tested, then those that prove to be LFs can be R‐combined to construct a more informative and richer LF. Note that the LEDA associated to the composed LF is always larger than (or in the worst case as large as) the single LEDAs. Therefore the estimation can only increase its accuracy.

A second advantage of R‐compositions is that the level sets of the union have the same shape of the union of the regions. The conditions under which an invariant set containing the origin in its interior can “shape” a Lyapunov function have been investigated in the literature, see for instance the survey paper [16]; special results have been obtained for the case of Linear Time Invariant (LTI) systems and compact, convex and contractive invariant sets. In contrast, this paper, based on the use of R-functions for the arbitrary composition of subsets of the RAS, may lead to the analysis of (generally) not convex sets.

The paper is organised as follows: next section describes R-functions, as they are expected not to be popular within the control community. Section III describes a systematic approach to use R-functions to compose LFs and the geometrical effects of the composition. Section IV provides an example of a classic RAS estimation problem. In the last section we conclude the paper and summarise our findings.

In the following, focus will be on time-continuous autonomous dynamical systems where the origin is an asymptotically stable equilibrium. The approach can however be extended without significant difficulties to include the case of other equilibrium points and deterministic inputs.

II. R-FUNCTIONS AS DIFFERENTIABLE LOGIC OPERATIONS

This section presents the use of R-functions as they will be used as the basic tool to compose Lyapunov functions. R-functions were first introduced by Rvachev about forty years ago, see [14] and [15], but their diffusion in countries outside Russia is very recent and only in the last few years R-functions have been used in practice, for instance in mechanical engineering applications. Here only the basic notions of R-functions and the properties that will be used for our purposes will be presented, while the full account of their theory, which goes beyond the purposes of this work, can be found in [17] or [18] and the references therein. In particular, here the same notation of [18] will be used.

The basic notion is that of logically charged real functions, which are functions of real variables having the property that their signs are completely determined by the signs of their arguments. A simple example of a logically charged real function is

\[
W(x, y, z) = x \cdot y \cdot z,
\]

where it is possible to build a truth table which establishes the sign of \( W \) as a function of the signs of the real arguments \( x, y \) and \( z \).

The connection between real valued function and boolean function is made by using the Heaviside function \( S_2 \) defined as

\[
S_2(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x \geq 0
\end{cases}
\]

Definition: (From [18]) A function \( f_{\Phi} : \mathbb{R}^n \rightarrow \mathbb{R} \) is an R-function if there exists a binary logic function \( \Phi : \mathbb{B}^n \rightarrow \mathbb{B} \), where \( \mathbb{B} = \{0, 1\} \) satisfying the commutative diagram of figure 1, where \( S_2^n : \mathbb{R}^n \rightarrow \mathbb{B}^n \) is the extension of the Heaviside function in the vector case.

Informally, a real function \( f_{\Phi} : \mathbb{R}^n \rightarrow \mathbb{R} \) is an R-function if it can change its property (sign) only when some of its arguments change the same property (sign). Therefore the commutative diagram of figure 1 implies that given an arbitrary logic function \( \Phi \), it is possible to find an equivalent representation with R-functions. Heaviside functions allow one to go from real valued functions to Boolean functions. Summarising, R-functions replace the logical and set-theoretic constructions with the corresponding real-valued functions.
A system of basic logic functions is complete if all logic functions can be constructed using the basic functions. Similarly a system of R-functions is sufficiently complete if its corresponding system of logic companion functions is complete (theorem from [14]). A sufficiently complete system of R-functions is not possible among polynomials. It can be proved that this task is impossible using only algebraic or analytic functions, therefore additional operations like square roots will be required. The choice of the system is not unique; in Table (I) a possible choice for R-negation, R-union and R-intersection is outlined, and it is the same that will be used in the remainder of this work. In particular, the choice of the real parameter $\gamma$ is performed iteratively, in analogy with usual Boolean logic equations.

From a geometrical point of view, if $f_1(x)$ and $f_2(x)$ are functions that are positive inside a geometrical region, and negative outside, then the R-intersection and the R-union represent the intersection and the union between sets. To avoid ambiguity due to the geometric correspondence between functions and sets, additional notation is added so that if $R(x)$ is an R-function, then in the remainder of this work $\hat{R}$ will refer to the set determined by $R(x)$, that is

$$\hat{R} = \{x \in \mathbb{R}^n | R(x) > 0\}$$

For instance, if $f_1(x) = 1 - x^T P_{1} x$ and $f_2(x) = 1 - x^T P_{2} x$, where $P_{1}^{-1} = \text{diag}(1, 9)$ and $P_{2}^{-1} = \text{diag}(9, 1)$, then the intersection and the union between $f_1$ and $f_2$ are represented in figure 2. Both the union and the intersection between two ellipsoids are described by equations that have positive values inside the region, negative values outside, and are zero on the boundary. Arbitrary unions and intersections between regions can be computed by applying the equations in Table (I) iteratively, in analogy with usual Boolean logic equations.

**Remark:** Note that functions for intersections and unions are not differentiable at the points where $f_1(x)$ and $f_2(x)$ are zero at the same time. In the example with ellipsoids in figure 2 this occurs in the intersections between ellipsoids. However, if differentiability is desired, functions can be upgraded to be of class $C^m$ by adding a new factor:

$$\left( f_1(x)^2 + f_2(x)^2 - \gamma f_1(x)f_2(x) \right)^{m/2}$$

The additional factor is always positive and therefore does not affect the logical properties of the function. Besides, the function is now differentiable $m$ times when each term is zero, with vanishing derivatives.

Theory of R-functions is much more than what has just been described until now, but the previous concepts are enough to show how to compose regions in the state space and obtain richer and less conventional LF.

### III. R-COMPOSITION OF Lyapunov functions

This section introduces a novel algorithm to estimate the RAS of a continuous-time autonomous dynamical system described by

$$\dot{x} = g(x),$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$ and the assumption that the origin of (5) is an asymptotically stable equilibrium. The following approach computes the union of regions of the state space, each one of which is known to be asymptotically stable.

**RAS estimate algorithm**

**Step 1:** $i = 1$

Solve the Lyapunov equation $A^T P_1 + P_1 A = -Q_i$ for a random choice of a positive definite matrix $Q_i$, where the transition matrix $A$ is the first order approximation of the Taylor expansion of $g$ in (5).

**Step 2:**

Solve an optimisation problem to find the LEDA associated to $P_i$ (and so derived from the particular choice of $Q_i$ performed in the previous step). Also notice that this corresponds to
find the largest ellipsoidal set where the quadratic Lyapunov function $V_i(x) = x^T P_i x$ is positive definite and its derivative is negative definite [19]-[20]. If $\{x \in \mathbb{R}^n | x^T P_i x < \alpha_i^*\}$ is the LEDA, we have, according to the notation introduced in II, that $R_i(x) = \alpha_i^* - x^T P_i x$ and $R_i = \{x \in \mathbb{R}^n | R_i(x) > 0\}$.

**Step 3:** $i := i + 1$

Go back to step 1 and iterate as long as $i \leq N$, where $N$ is the desired number of Lyapunov functions one is interested in composing.

**Step 4:**
Compute the R-composition of $R_1(x), R_2(x), \cdots, R_N(x)$, which corresponds equivalently to the geometrical union of the single regions as

$$\hat{R}_{\cup} = \hat{R}_1 \cup \hat{R}_2 \cup \cdots \cup \hat{R}_N,$$

according to the analytical equation for the union, as in Table (I). Also notice that the union composition can be performed iteratively, that is $\hat{R}_{\cup} = \left(\cdots (\hat{R}_1 \cup \hat{R}_2) \cup \hat{R}_3 \cup \cdots \right) \cup \hat{R}_N,$ as a consequence of the associative properties of the Boolean union function which still hold in the function case.

Although $\hat{R}_{\cup}$ is obviously inside the RAS and represents the best estimate of the RAS that can be done on the basis of the single LEDAs, it still needs to be checked whether the correspondent R-composition $R_{\cup}(x)$ still gives rise to a LF. This question requires further discussion, and the solution might depend on the particular choice of $\gamma$ inside the union function from Table (I) as it is clarified in the next subsection.

### A. R-composition as a Lyapunov function

In the previous paragraph the R-composition of Lyapunov function is introduced, and the main equation to compute the union of two function is provided by

$$R_{\cup}(x) = R_1(x) + R_2(x) + \sqrt{R_1(x)^2 + R_2(x)^2 - \gamma R_1(x)R_2(x)},$$

with $\gamma \in [0, 2]$, as introduced in II. Although the sign of $R_{\cup}(x)$ is independent from the choice of $\gamma$, because it is positive inside the union region, zero on the boundary and negative outside, the inner level curves are affected by the choice of $\gamma$, as it is illustrated in figure 3 for the extreme cases $\gamma = 0$ and $\gamma = 2$. The case of $\gamma = 2$ corresponds to the situation of all the level curves homotopic to the union boundary, while inner level curves are smoothed by a smaller choice of $\gamma$. The case of $\gamma = 2$ has been widely studied in the literature (see for example the recent reference [21]) because the union function always gives rise to a Lyapunov function. In fact, given more Lyapunov functions, the R-composition in the case $\gamma = 2$ corresponds to

$$V(x) = \max_i \{V_i(x)\}$$

More precisely, in [22] it was pointed out that this special form of functions should be called Lyapunov-like function, since the classical Lyapunov function condition that $V(x)$ should be continuously differentiable is relaxed. On the other hand, if the choice of smaller values of $\gamma$ is preferred, the function gets differentiable everywhere in $\hat{R}_{\cup}$, although now the corresponding $R_{\cup}(x)$ is not guaranteed to be always a Lyapunov function. The special case of $\gamma = 0$ was first analysed in [23], while here sufficient conditions for the union function to give rise to a LF are given for the general case $\gamma \in [0, 2]$.

In the remainder of this section the following sufficient condition for the definition of a LF is used:

**Lemma (5.7.4 from [24]):** Let $\Sigma$ be a continuous-time system and $V : X \rightarrow \mathbb{R}$ a continuous function. Assume that $O \subset X \subset \mathbb{R}^n$ is an open subset for which the restriction of $V$ to $O$ is continuously differentiable, $V$ is proper at the equilibrium and $V$ is positive definite in $O$. Then, a sufficient condition for $V$ to be a Lyapunov function is that for each $x \in O, x \neq x^0$, then $\nabla V(x) \cdot g(x) < 0$,

where $x^0 \in O$ is the equilibrium point in the general case and the dynamical system $\Sigma$ is described by $\dot{x} = g(x)$ as in (5).

**Theorem 3.1:** Assume $V_1 : \hat{R}_1 \rightarrow \mathbb{R}$ and $V_2 : \hat{R}_2 \rightarrow \mathbb{R}$, with $\hat{R}_1, \hat{R}_2 \subseteq \mathbb{R}^n$, are two Lyapunov functions for the dynamical system (5), then a sufficient condition for the function $-R_{\cup}(x) + R_{\cup}(0)$ to be a Lyapunov function is that $\nabla V_1(x) \cdot g(x) < 0$ and $\nabla V_2(x) \cdot g(x) < 0$ for all $x \in \hat{R}_{\cup} = \hat{R}_1 \cup \hat{R}_2$, for every choice of $\gamma \in [0, 2]$.

**Proof:** Let us call $\alpha_i^*$ the maximum value of $V_i(x)$ in $\hat{R}_i$ (roughly speaking this corresponds to the largest level curve...
and to the fact that $R_i(0) > 0$ in $\tilde{R}$, and it total derivative has opposite sign with respect to $V_i(x)$ and thus is positive in the whole $\tilde{R}$ by hypothesis. Therefore $-R_{ij}(x) + R_{ij}(0)$ is a positive definite function. In particular by substitution in (7), $R_{ij}(0) = \alpha_i^* + \alpha_j^* + \sqrt{V_1(x)} - \gamma \alpha_i^* \alpha_j^*$. Note that $R_{ij}$ is differentiable everywhere for $\gamma \in [0, 2]$ in $\tilde{R}$, since $R_{ij}$ is an open set. The total derivative $\nabla R_{ij}(x) \cdot g(x)$, where the dependence on the parameter $\gamma$ is made explicit, is

$$\nabla R_{ij}(x) \cdot g(x) = \frac{\nabla R_1(x) \cdot g(x) \left(2\sqrt{V_1(x)} + R_2(x) - \gamma R_1(x) R_2(x) + 2R_1(x) - \gamma R_2(x)\right) + \nabla R_2(x) \cdot g(x) \left(2\sqrt{V_1(x)} + R_2(x) - \gamma R_1(x) R_2(x) + 2R_1(x) - \gamma R_2(x)\right)}{2\sqrt{V_1(x)} + R_2(x) - \gamma R_1(x) R_2(x)}.$$  

(9)

Since $\nabla R_1(x) \cdot g(x)$ and $\nabla R_2(x) \cdot g(x)$ are positive in $\tilde{R}$, by assumption, we want to prove that the terms in brackets are positive as well in the whole region $\tilde{R}$ so that the positive definite function $-R_{ij} + R_{ij}(0)$ has its total derivative negative as required [20].

The first term inside the brackets is

$$\sqrt{V_1(x)} + R_2(x) - \gamma R_1(x) R_2(x) + 2R_1(x) - \gamma R_2(x),$$  

while the second term is equivalent with reversed $1-2$ indices. The term (10) is always greater or equal than zero for each possible sign of $R_1$ and $R_2$, thanks to the triangle inequality and to the fact that $\gamma \in [0, 2]$.

Remark 1: The thesis of the previous theorem was that $-R_{ij}(x) + R_{ij}(0)$ is a Lyapunov function provided certain assumptions hold. With a slight abuse of notation, one can simply say that $R_{ij}(x)$ is a Lyapunov function itself. The abuse consists in the fact that $R_{ij}(x)$ is a negative definite function (plus a displacement) with positive total derivative in the union set and therefore $R_{ij}(x)$ is a LF with reversed signs with respect to the usual definition of a LF. The reversed sign notation is however particularly convenient for our purposes since the union R-function can now be directly considered a LF.

Remark 2: Note that theorem 3.1 is only a sufficient condition and its convenience is constrained by the fact that it both depends on the dynamical system in exam and the particular choice of the initial Lyapunov functions.

Theorem 3.2: Assume $V_1(x)$ and $V_2(x)$ are two quadratic Lyapunov functions for the dynamical linear systems $\dot{x} = Ax$. Then the function $-R_{ij}(x) + R_{ij}(0)$ is always a Lyapunov function in $\tilde{R}_i$, for every choice of $\gamma \in [0, 2]$.

Proof: Let us consider the two quadratic Lyapunov functions $V_1(x) = x^T P_1 x$ and $V_2(x) = x^T P_2 x$. It is well known that the two Lyapunov equations $A^T P_1 + P_1 A = -Q_1$ and $A^T P_2 + P_2 A = -Q_2$ are then satisfied for two opportune positive definite matrices $Q_1$ and $Q_2$. In the case of quadratic Lyapunov functions, according to the usual notation of LEDAs, $R_i(x) = \alpha_i^* - x^T P_i x$. In (9), the Ly derivative terms for the special case in exam are $\nabla R_i(x) \cdot g(x) = -x^T (A^T P_i + P_i A) x = x^T Q_i x$, where $Q_i$ are positive definite matrices for $i = 1, 2$ as previously stated. Therefore the Lie derivative terms are positive, which corresponds to the sufficient condition of theorem 3.1 and so $-R_{ij}(x) + R_{ij}(0)$ is a Lyapunov function.

Remark 3: It is known from literature, for instance [16], that in the case of LTI systems, if $P_1$ and $P_2$ are two positive definite matrices that satisfy the correspondent Lyapunov equations $A^T P_i + P_i A < 0$, for $i = 1, 2$, than also the convex combination $\alpha P_1 + (1-\alpha) P_2$ does. The difference with our theorem 3.2 is that here a special case of non-convex composition of $P_1$ and $P_2$ is investigated. The level curves of a star-shaped Lyapunov function are shown in figure 3 for two different choice of $\gamma$.

IV. Examples

The following nonlinear dynamical system had already been studied in [25]

$$\begin{cases} x_1' = -x_1 + 2x_1^3 x_2^2 \\ x_2' = -x_2 \end{cases}$$

(11)

The exact RAS is known to be the set $\{ x \in \mathbb{R}^2 \mid x_1^2 x_2^2 < 1 \}$.

In [25] a first LF $V_1(x) = \frac{x_1^2}{1+x_1^2} + 2x_2^4$ is proposed and here we compose it with a second LF $V_2(x)$, in order to obtain a non conventional LF and a more accurate RAS estimate. The non quadratic function $V_1$ is more stretched horizontally along the component $x_1$, so we choose the vertical quadratic Lyapunov function $V_2(x) = x^T P_2 x$, with $P_2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$.

First of all we find the LEDA associated to $P_2$. Let $R(x)$ be the R-composition of $V_1$ and $V_2$ and $\tilde{R}$ the corresponding RAS estimate as shown in figure 4. In this example $\tilde{R}_1(x)$ and $\tilde{R}_2(x)$ are always positive inside the union region $\tilde{R}$, so $R(x)$ is always a LF for every choice of $\gamma \in [0, 2]$.
represents a first step on the systematic use of R-functions to arbitrarily compose LFs to obtain a richer family of LFs. Here, a general purpose algorithm was presented, while some future research will focus on the application of the same approach to a special class of dynamical systems, such for instance the popular linear constrained ones.

**REFERENCES**


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**V. Conclusion**

This paper provides a class of R-functions to compose LFs to obtain a more flexible and richer function associated to a RAS estimate that is the geometrical union of single LEDAs. A parameter $\gamma$ affects the smoothness of the union function, and the special choice of $\gamma = 2$ guarantees that the union function still corresponds to a LF. However, in some applications it is more convenient to choose other values of $\gamma$ to add the differentiability property of the union function, so sufficient conditions for the R-composition to be still a LF in the general case $\gamma \neq 2$ are presented. The proposed approach does not require any *a priori* information about the RAS, although a clever choice of single LFs clearly improves the final estimate. However, even if the single LFs are chosen in a poor way and their associated LEDAs are very small in the phase space, still their union represents an improved estimate. This paper

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**Fig. 5.** Level curves of the R-composition with $\gamma = 2$. The trajectories of the state diverge according to the unstable initial conditions or evolve in accordance to the Lyapunov-like function.

**Fig. 6.** Level curves of the R-composition with $\gamma = 0$. Figures 5 and 6 show the level curves of $R(x)$ with $\gamma = 2$ and with $\gamma = 0$, and trajectories of the state according to dynamics (11). The initial conditions are chosen inside a square of side 4.