R-composition of quadratic Lyapunov functions for stabilizability of linear differential inclusions

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Abstract:
A novel state feedback control technique to stabilise linear differential inclusions through composite quadratic Lyapunov functions is presented. By using a gradient-based control technique, the minimum effort control is composed through intersection and union operations, derived from the theory of R-functions. While conventional min and max compositions are recovered as a special case, it is shown that smoother sublevel sets and everywhere differentiability are obtained tuning the composition parameter. Examples of both intersection and union compositions are provided to show that intermediate control performances in terms of convergence time are obtained, while improved performances in the control signal can be achieved.

Keywords: Nonquadratic Lyapunov functions, quadratic stabilizability.

1. INTRODUCTION

When designing a state feedback control for a real dynamical system, one should take into account possible uncertain parameters in the state equations. Uncertainties can be modeled using switching systems in which system dynamics are allowed to switch among a set of possible ones [1]. Asymptotic stability of switching systems is intended under arbitrary unknown switching sequence and it is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems [2]. Therefore the stabilizability problem of linear systems under an arbitrary switchings can be formulated as the stabilizability problem of a particular linear time-variant system [3], [4].

Linear Differential Inclusions (LDIs) provide an alternative way of modeling systems affected by nonlinearities and time-variant uncertainties. The classic solution for the stabilisation problem of LDIs is based on the use of a Lyapunov Function (LF). An uncertain system which admits a stabilizing state feedback control associated to a quadratic LF is said to be quadratically stabilizable. In particular, necessary and sufficient conditions under which an uncertain system can be quadratically stabilised by a linear feedback controller are well known in the literature [5].

Although commonly accepted, linear quadratic stabilisation is conservative since there are stable uncertain systems which are quadratically stabilizable via nonlinear control but not quadratically stabilizable via linear feedback control [6].
improve the estimate of the Region of Asymptotic Stability (RAS) [19], [20]. As a consequence, non conventional, star convex RAS estimates were obtained, in contrast with classic convex results [18].

The paper is organised as follows: next section shortly describes R-functions. Section 3 illustrates the use of R-functions as differentiable CLFs with the use of the Minimum Effort Control (MEC), thus obtaining a continuous law [22]. Section 4 provides examples of some control problems of linear systems with time-variant uncertainties. In the last section we conclude the paper and summarise our work.

2. R-FUNCTIONS

2.1 Theoretical background

This section describes R-functions as they will be used as the basic tool to compose quadratic functions. Here only the basic notions of R-functions and the properties useful for our purposes will be presented. A full account of their theory can be found in [16]. Here, the same notation already introduced in [20] is used.

R-functions are logically charged real functions, i.e. functions of several real variables having the property that their signs are completely determined by the signs of their arguments. The connection between real valued functions and Boolean functions is made by using the Heaviside function $S_2$ defined as

$$S_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$  

Definition 1. (from [17]) A function $f_\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an R-function if there exists a binary logic function $\Phi : \mathbb{B}^n \rightarrow \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$ satisfying the commutative diagram of Figure 1, where $S_2^\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the extension of the Heaviside function in the vector case.

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{f_\Phi} & \mathbb{R} \\
S_2^n & \downarrow & S_2 \\
\mathbb{B}^n & \xrightarrow{\Phi} & \mathbb{B}
\end{array}$$

Fig. 1. Commutative diagram for an R-function

Informally, a real function $f_\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an R-function if it can change its property (sign) only when some of its arguments change the same property (sign).

The representation of geometrical operations such as intersection and union of subsets is not possible using only polynomial functions [14], therefore additional functions as $p$-roots are required [15]. For example possible composition rules that exploit square roots, respectively of negation, intersection and union, are the following:

$$f_\cap = \frac{1}{2 - \sqrt{2 - 2\alpha}} \left( f_1 + f_2 - \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2} \right)$$

$$f_\cup = \frac{1}{2 + \sqrt{2 - 2\alpha}} \left( f_1 + f_2 + \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2} \right).$$

where all functions $f_1, f_2, f_\cap, f_\cup, f_\alpha^\cap, f_\alpha^\cup : \mathbb{R}^n \rightarrow \mathbb{R}$ and parameter $\alpha \in [0, 1] \subset \mathbb{R}$. To improve clarity, in the union and intersection operations dependence from $\alpha$ is indicated explicitly. The multiplicative factor $\frac{1}{2 \pm \sqrt{2 - 2\alpha}}$ normalises the composed R-function which is equal to 1 in the points where both $f_1$ and $f_2$ are equal to 1. The normalisation factor is always positive and therefore it does not affect the sign properties of the composed function. Equations (2) are valid composition rules because it is always possible to establish the sign of $f_\cap, f_\cup, f_\alpha^\cap$ and $f_\alpha^\cup$ by just knowing the signs of $f_1$ and $f_2$ accordingly to classic Boolean rules. This follows from applying the law of cosines and the triangle inequality to the triangle of sides $f_1, f_2$ and $\sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2}$, in the case that $\alpha$ is the cosine of the angle included between $f_1$ and $f_2$.

It is also convenient to add the notation $\hat{f}$ as the subset of $\mathbb{R}^n$ where function $f$ is strictly positive.

$$\hat{f} = \{ x \in \mathbb{R}^n : f(x) > 0 \}$$

We will only consider functions whose associated $\hat{f}$ is a connected set.

Remark 1. Note that functions for intersections $f_\cap^\alpha$ and unions $f_\cup^\alpha$ are differentiable, respectively in $f_\cap$ and $f_\cup$, $\forall \alpha \in (0, 1)$.

Example. Consider two quadratic functions $x^T P_1 x$ and $x^T P_2 x$ in $\mathbb{R}^2$. Let us consider each of the two functions in the region where their values are smaller or equal to one.

$$f_i(x) = 1 - x^T P_i x, \quad \hat{f}_i = \{ x \in \mathbb{R}^2 : x^T P_i x \leq 1 \}, \quad i = 1, 2.$$ 

By applying equations (2), compute $f_\cap^\alpha(x) = f_1(x) \cap f_2(x)$ and $f_\cup^\alpha(x) = f_1(x) \cup f_2(x)$. As a consequence of the commutativity between sets and functions, also $f_\cap = f_1 \cap f_2$ and $f_\cup = f_1 \cup f_2$. It is evident that the bounding set $f_\cap (\hat{f}_1 \cap \hat{f}_2)$ is independent from $\alpha$, as the composite function is positive inside the intersection (union) region, zero on the boundary and negative outside. However, level curves are clearly affected by the choice of $\alpha$. They become progressively smoother by reducing the value of the shape parameter $\alpha$ and in the case $\alpha = 0$ they are the smoothest for the square root law expressed in equations (2). Figure 2 illustrates the level curves of the intersection and the union of two different ellipsoidal R-functions, in the nonsmooth limit case of $\alpha = 1$ and in the case $\alpha = 0$. The matrices $P_i$ are: $P_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Remark 2. R-functions can be used to compose general functions and not just quadratic ones. Examples of different compositions for state feedback control applications can be found in [13].

Although theory of R-functions is much more than what has just been described until now, the previous concepts on R-composition are enough for the purposes of this work.
\[ F(x(t)) \in \text{conv} \left\{ Ax(t) : A \in \mathcal{A} \right\}, \]
\[ \mathcal{A} = \left\{ \sum_{i=1}^{m} \mu_i A_i : \mu_i \in [0, 1] \forall i, \ \sum_{i=1}^{m} \mu_i = 1 \right\} \]
and \( \text{conv} \{ \cdot \} \) denotes the convex hull. Alternatively, the same problem can be stated with a slightly different notation (7) which is more typical of polytopic uncertain linear time-variant systems.

As already mentioned, although it might look surprising, studying the stability of systems (6) or (7) corresponds to studying the stability of a switching system where the state \( x(t) \) and the controlled inputs \( u(t) \) belong to \( \mathbb{R}^n \) and \( \mathbb{R}^q \) respectively. The uncertain time-variant parameter \( \mu(t) \in \mathbb{R}^m \) is a piecewise continuous function. The system matrix \( A(\mu(t)) \) is constrained to belong to the matrix polytope

\[ A(\mu(t)) = \sum_{i=1}^{m} A_i \mu_i(t) \]

where

\[ \mu(t) \in \mathcal{M} = \left\{ \mu \in \mathbb{R}^m_+ : \sum_{i=1}^{m} \mu_i = 1 \right\}. \]

As already mentioned, although it might look surprising, studying the stability of systems (6) or (7) corresponds to study the stability of a switching system where the state matrix is allowed to vary among the set of \( A_i \) [2].

### 3.1 R-composition of quadratic Lyapunov functions

The aim of this work is to investigate the proprieties of the R-intersection and the R-union of \( m \) quadratic Lyapunov functions \( x^T P_i x \) for different values of the composition parameter \( \alpha \). Moreover the control performances of the R-compositions are compared with the same control strategy applied with single quadratic LFs.
Given the $n \times n$ matrices $P_1, \ldots, P_m$, the normalised R-functions $R_i$, for $i = 1, \ldots, m$, are
$$R_i(x) = 1 - x^\top P_i x.$$ (10)
The R-intersection is the function
$$R_i^\cap = R_i \cap \ldots \cap R_m$$ (11)
and the R-union is the function
$$R_i^\cup = R_i \cup \ldots \cup R_m,$$ (12)
according to the composition rules (2).

Note that $R_i^\cap(x) > 0 \forall x \in \bigcap_{i=1}^m \hat{R}_i$ and $\max_{x \in \hat{R}_i} R_i^\cap(x) = R_i^\cap(0) = 1$, therefore the associated candidate LF is
$$V_i^\cap = 1 - R_i^\cap.$$ (13)
The change of sign and the $+1$ are required to recover the classic definition of a candidate LF (see [19] for further details).

Similarly $R_i^\cup(x) > 0 \forall x \in \bigcup_{i=1}^m \hat{R}_i$, $\max_{x \in \hat{R}_i} R_i^\cup(x) = R_i^\cup(0) = 1$ and the associated candidate LF is
$$V_i^\cup = 1 - R_i^\cup.$$ (14)

### 3.2 Minimum effort control

A CLF $V$ with speed of convergence $\beta$ (also called contractivity factor) must satisfy
$$\nabla V(x) \dot{x} \leq -\beta V(x)$$ (15)
where the gradient $\nabla V$ is considered a row vector. Equation (15) is guaranteed by control signals $u$ such that
$$\nabla V(x) Bu(x) \leq -\beta V(x) - \max \nabla V(x) A x.$$ (16)
The set of feasible controls of (16) is convex for each $x$ since it is the intersection of hyperplanes. Therefore, provided that such set is not empty, it is possible to choose the control with minimum 2-norm which is known in literature as minimum effort control [21]. The explicit expression of the control is
$$u(x) = -\max \left\{ 0, \frac{\beta V(x) + \max_{y} \nabla V(x) A x}{\nabla V(x) B y} \right\} \nabla V(x) B.$$ (17)

According to [22], the minimum effort control is continuous if the CLF $V$ is everywhere differentiable (except at the origin). For this reason, it is convenient to decrease the value of the shaping parameter $\alpha$ so that differentiability is gained, and a continuous control signal is obtained with a gradient-based strategy (17).

Both $V_i^\cap$ and $V_i^\cup$ have to be proved to be CLFs. The general case for $\alpha \in [0, 1]$ can be established analytically thanks to the sufficient condition presented in [21] (Theorem 5.1). This theorem states that a sufficient condition for stabilizability of systems (6) or (7) with the Lyapunov function $V$ and bounding function $c_0 = \beta V$ is
$$\phi(x) < 0 \ \forall x \in \mathcal{N},$$ (18)
where
$$\phi(x) \triangleq \max_i \{ \nabla V(x) A_i x \} + c_0(x)$$ (19)
and
$$\mathcal{N} \triangleq \{ x \in \mathbb{R}^n : \nabla V(x) B = 0 \}.$$ (20)
The previous theorem can be used as an effective analytical tool to prove that the smoothing with a particular value of $\alpha$ still gives rise to a CLF.

In the special case of $\alpha = 1$, it is sufficient to know that condition (18) is satisfied for the single LFs to guarantee that the composed function is a CLF as well, according to the following theorem.

**Theorem 1.** Assume $V_1, \ldots, V_m$, with $V_i : \hat{R} \to \mathbb{R}$ for $i = 1, \ldots, m$ satisfy the sufficient condition (18). Then also the composed LFs $V_{\cap}^\cup$ and $V_{\cup}^\cap$ satisfy the same condition.

**Proof.** Regarding the intersection composition, in a particular state point $y \in \mathbb{R}^n$, $\nabla V_{\cap}^\cup(y) = \nabla V_{\cap}(y)$, where $V_{\cap}(y) = \max_i \{ V_i(y) \}$. Therefore the set (20) of the composed LF is a subset of the set (20) of one of the single LFs, for which condition (18) held by hypothesis. Proof that also the union composition $V_{\cup}^\cap$ satisfies condition (18) follows the same argument with the max operator substituted by the min one.

**Remark 3.** Theorem 1 does not hold in the reverse direction, i.e. the composed LF might satisfy condition (18), while the single LFs do not. This fact confirms that it is more likely to find the solution within the set of composed functions.

### 4. EXAMPLES

This section shows a comparison among the composite quadratic functions in terms of control performances. The use of composite quadratic functions is motivated by the fact that they are a universal class of functions for the stabilizability problem of LDIs and the common CQLFs are a special case of composite quadratic functions.

**Example 1.** System [27]

$$
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u(t)
$$ (21)

has to be stabilised for all bounded uncertainties $|\mu(t)| \leq 0.9$. The associated LDI is characterized by the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1.9 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -1 \end{bmatrix}.$$ (22)

**Remark 4.** Although each system $\ddot{x} = A_i x$ is asymptotically stable, in [27] it is shown that the open loop uncertain system cannot be quadratically stabilised by a linear state feedback controller. Therefore it is not possible to find a suitable CQLF exploiting LMI techniques.

We start from the quadratic CLFs for the MEC strategy, $V_i = x^\top P_i x$, $i = 1, 2$, where

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2.2627 & 0.5657 \\ 0.5657 & 1.1314 \end{bmatrix}.$$ (23)

The level sets of $V_1$ have a vertical shape in the state space plane, while the level sets of $V_2$ are at $45^\circ$. By applying Theorem 1, also $V_{\cap}^\cup$ and $V_{\cup}^\cap$ are suitable CLFs. Then one can successfully verify the condition (18) for $V_{\cap}^\cup$ and $V_{\cup}^\cap$, with $\alpha = 0$ and $\alpha = 0.5$, for a decay rate $\beta \leq 0.2$. A multiplicative feedback gain $\gamma = 4$ is added to the control law, in order to decrease the convergence time. In the simulations, the R-intersection $V_{\cap}^\cup$ and the R-union $V_{\cup}^\cap$ are scaled such that the corresponding regions $\hat{R}_{\cap}$ and $\hat{R}_{\cup}$
Table 1. Direct comparison of average $|u|_{\text{max}}$, IAU, IADU, ISE performance indices and convergence time $T$ over 100 random initial conditions $x_0$ inside the square of side $2$ centered at the origin. The control strategy is the MEC.

| Control LF | $|u|_{\text{max}}$ | IAU | IADU | ISE | $T$ |
|------------|----------------|-----|------|-----|-----|
| $V_1$      | 0.4725         | 0.1892 | 0.7168 | 0.4332 | 8.0076 |
| $V_2$      | 0.3493         | 0.1510 | 0.5222 | 0.5041 | 8.8121 |
| $\max(V_1, V_2)$ | 0.3529 | 0.1587 | 0.6846 | 0.4933 | 8.6592 |
| $\min(V_1, V_2)$ | 0.4723 | 0.1827 | 0.7126 | 0.4414 | 8.1523 |
| $V_1 \lor V_2$ | 0.3779 | 0.1574 | 0.5702 | 0.4896 | 8.6688 |
| $V_1 \land V_2$ | 0.3148 | 0.1215 | 0.5005 | 0.5483 | 9.8216 |
| $V_1 \lor_0 V_2$ | 0.3805 | 0.1576 | 0.5811 | 0.4896 | 8.6674 |
| $V_1 \land_0 V_2$ | 0.3184 | 0.1245 | 0.5051 | 0.5483 | 9.7275 |

In Table 1, values of typical control indices are shown: $|u|_{\text{max}}$ is the maximum value of the control signal $u$, and it should be small to avoid peaks of the control effort; IAU is the Integral of the Absolute value of the control signal $u$, and it should be small to reduce the average control effort; IADU is the Integral of the Absolute value of the Derivative on the control signal $u$, and it is desired to be small to avoid stress of the control actuator; ISE is the Integral of the Square value of the Error, and it should be small to avoid large errors or slow convergence; finally $T$ represents the required time of convergence ($2$—norm of the state vector smaller than $10^{-3}$).

Simulation results show that the control performances of the max and min compositions average the performances of the single quadratic CLFs when a MEC strategy is applied. The classic intersection achieves performances that are close to the ones of the best single quadratic CLF, while the classic union yields control indices close to the ones of the worst quadratic CLF. Moreover, smoothing the inner level curves of the max and min functions is particularly efficient for the control peak $|u|_{\text{max}}$ and the IADU index, because the smooth level curves decrease the fast variations of the control signal. Figure 3 and 4 show some state trajectories which are in accordance with the composite level curves, with the shape parameter $\alpha = 0.5$.

Example 2. The second example is the dynamical system taken from [7].

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \delta(t) \\ 0 \end{pmatrix} u(t)$$

with bounded uncertainty $|\delta(t)| \leq 0.5$. Although matrices $A_1$ and $A_2$ have a positive eigenvalue, in this case it is possible to find a CQLF.

Again we assume that two suitable quadratic CLFs $V_1 = x^\top P_1 x$ and $V_2 = x^\top P_2 x$ for the MEC state feedback strategy, with decay rate $\beta \leq 0.2$, are available, where

$$P_1 = \begin{bmatrix} 6.6637 & -4.2432 \\ -4.2432 & 4.6002 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}. $$

Therefore $V_1^\alpha, V_1^\beta$ are CLFs by means of Theorem 1. Besides, condition (18) is satisfied for $\alpha = 0$ and $\alpha = 0.5$ so also the corresponding $V_1^\alpha s, V_1^\beta s$ are suitable CLFs for the gradient-based control strategy.

The averaged simulation results have quite the same qualitative behavior of the previous example. In particular, the IADU indices of the R-functions are lower with respect to the IADU indices of the correspondent max and min functions. Figures 5 and 6 show some state trajectories and level curves with $\alpha = 0.5$. 

Table 2. Direct comparison of average $|u|_{\text{max}}$, IAU, IADU, ISE performance indices and convergence time $T$ over 100 random initial conditions $x_0$ inside the square of side 2 centered at the origin. The control strategy is always the MEC.

| Control LF | $|u|_{\text{max}}$ | IAU | IADU | ISE | $T$ |
|------------|----------------|-----|------|-----|-----|
| $V_1$      | 0.4639         | 0.1502 | 0.6137 | 0.2864 | 5.9721 |
| $V_2$      | 0.5751         | 0.1336 | 0.6506 | 0.2682 | 4.9255 |
| $\max(V_1, V_2)$ | 0.5072 | 0.1361 | 0.6458 | 0.2751 | 5.0858 |
| $\min(V_1, V_2)$ | 0.5430 | 0.1474 | 0.6897 | 0.2793 | 5.7780 |
| $V_1 \land_0 V_2$ | 0.5072 | 0.1424 | 0.6123 | 0.2771 | 5.5077 |
| $V_1 \lor_0 V_2$ | 0.4793 | 0.1478 | 0.5657 | 0.2844 | 5.7862 |
| $V_1 \lor_0 V_2$ | 0.5077 | 0.1424 | 0.6128 | 0.2771 | 5.5077 |
| $V_1 \lor_0 V_2$ | 0.4815 | 0.1473 | 0.5694 | 0.2844 | 5.7592 |
conditions that guarantee the composite function to be a CLF without testing condition (18).

REFERENCES


5. CONCLUSIONS

In the paper the smoothed intersection and union of quadratic functions is proposed to stabilise LDS. A state feedback gradient-based control is used together with a smooth CLF obtained as intersection and union of single quadratic LFs, within the R-functions framework.

The proposed approach is convenient because there exist uncertain systems that are not quadratically stabilizable by means of a linear controller. Whereas, the intersection and the union of quadratic functions constitutes a universal class for stabilizability of linear differential inclusions. The intersection and the union functions yield intermediate control performances with respect to the correspondent single control Lyapunov functions. The quadratic composite function is then smoothed to achieve everywhere differentiability so that gradient-based controllers provide continuous control signals and reduce the fast variations of the control signal. This solution generalises the classic Lyapunov-like approach in which the max and min operators drop the differentiability condition.

The drawback of the gradient-based control law is that, in general, the sufficient condition (18) has to be verified for each single control problem, because the presented theorem is valid only for the limit value of the shape parameter. A future line of research will focus on searching analytical...


