Abstract—Given two control Lyapunov functions (CLFs), a “merging” is a new CLF whose gradient is a positive combination of the gradients of the two parents CLFs. The merging function is an important trade-off since this new function may, for instance, approximate one of the two parents functions close to the origin while being close to the other far away. For nonlinear control-affine systems, some equivalence properties are shown between the control-sharing property, i.e., the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of the two given CLFs, and the existence of merging CLFs. It is shown that, even for linear time-invariant systems, the control-sharing property does not always hold, with the remarkable exception of planar systems. The class of linear differential inclusions is also discussed and similar equivalence results are presented. For this class of systems, linear matrix inequalities conditions are provided to guarantee the control-sharing property. Finally, a constructive procedure, based on the recently-considered “control Lyapunov R-functions”, is defined to merge two smooth positively homogeneous CLFs.

Index Terms—Composite control Lyapunov functions; stabilizability of linear differential inclusions.

I. INTRODUCTION

Control design must quite often compromise among performance, robustness and constraints, and Lyapunov theory offers suitable tools in this regard. The essential goals of the constrained robust performance control design are assuring stability, fulfilling constraints and facing uncertainties. Lyapunov-based techniques for constrained robust control trace back to the 70s [1]. The solutions originally proposed where based on quadratic Lyapunov functions [2] and linear (possibly saturated) controllers. However it became immediately clear that quadratic functions are quite conservative in terms of both domain of attraction (DoA) [3], [4] and robustness margin [5]. Solutions based on non-quadratic Lyapunov functions have been suggested for constrained control, initially based on the polyhedral ones [3], [4] or smoothed-polyhedral functions [6]. An intensive research activity has then been devoted in discovering suitable classes of Lyapunov functions, including the composite Lyapunov functions [7], truncated quadratic functions [8], [9], [10] and polynomial homogeneous functions [11], [12]. Surveys can be found in [13], [14].

There is a fundamental issue in the Lyapunov-based approach for control in which constraints, robustness and optimality are of concern: it turns out that a single Lyapunov function is typically suitable for one of these goals, but often ineffective for the others. For instance the size of the “safe set”, namely the domain of initial conditions for which the constraints are not violated, can be quite large if we consider a particular Lyapunov function. On the contrary, a different Lyapunov function based on some “optimal” cost function and assuring local “optimality”, may provide a significantly smaller domain. The established solution to this problem is the control switching strategy. Two controllers are designed, each associated with one of these functions, whose domains of attractions are typically (not necessarily) nested. The control system switches from the “external” to the locally optimal gain as long as the state reaches the “smaller” region of attraction. Obviously, several control gains can be considered with several controlled-invariant regions [15], [16].

The drawback of the scheme is the discontinuity which can be “dangerous”, since the system state and the control could be subject to jumps which can be even be persistent in the presence of noise. Therefore it is of interest to find ways to “merge” the two Lyapunov functions in order to have a “smooth” transient from the level set of the “external” one to the “internal” one. We refer to a procedure of this kind as merging.

Recently, Andrieu and Prieur [17] proved that it is possible to merge two Control Lyapunov Functions (CLFs), in a setting actually related to the problem of uniting local and global controllers [18], [19]. Their technique works under the assumption that there exists a suitable domain in which the two control Lyapunov function share a common control [17, Proposition 2.2]. More recently, Clarke [20] showed how to solve the problem of merging two semiconcave (continuous, locally Lipschitz but not everywhere-differentiable) CLFs, deriving a semiconcave function based on the min operator.

In this work we investigate the control-sharing property, namely the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of two given Lyapunov functions. We show some equivalence properties about the control sharing and the possibility of adopting a merging procedure.

The control-sharing property is not necessarily satisfied even for linear systems, with the remarkable exception of the planar case (i.e., with two-dimensional state space). Therefore, we provide efficient computational tests to check the control-sharing property for some special classes of functions including polyhedral, quadratic, piecewise quadratic and truncated ellipsoids.

Finally we provide as merging example the technique based on the “R-functions” theory, first presented in [21], [22], and
we show how local optimality can be compromised with a large DoA, under constraints, adopting a single smooth function.

The essential results of the paper are summarized next.

- For planar linear time-invariant systems two convex CLFs always share a control. A third-order counterexample shows that this is not true in general.

- Given two CLFs $V_1, V_2$, a merging function $V$ is defined as any positive definite function whose gradient has the form $\nabla V(x) = \gamma_1(x)\nabla V_1(x) + \gamma_2(x)\nabla V_2(x)$, where $\gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous functions. For the class of control-affine nonlinear systems, it is shown that any merging function $V$ (i.e. for any possible $\gamma_1$ and $\gamma_2$) is also a CLF if and only if $V_1$ and $V_2$ share a stabilizing control.

- For the class of linear systems, the above statements are also equivalent to the existence of a “regular” type merging, namely, the case in which $\nabla V$ is “close” to $\nabla V_1$ far from the state-space origin and $\nabla V$ is “close” to $\nabla V_2$ in a neighborhood of the origin.

- Several conditions are provided to check the control-sharing property. These are based on Linear Programming (LP) in the case of piecewise-linear functions and Linear Matrix Inequalities (LMIs) in the case of piecewise-quadratic and truncated-ellipsoidal functions.

- The “R-composition” merging technique presented in [23] is considered to solve the problem of preserving the large DoA under constraints of one Lyapunov function and assuring local optimality guaranteed by the other at the same time.

### A. Notation

$I_n$ denotes the $n \times n$ identity matrix. $T_+ := (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$. The notation co($\cdot$) denotes the convex hull [24]. int$S$ denotes the interior of a set $S$ and $S^\circ$ denotes its boundary. For any positive (semi)definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $L_V$ denotes its 1-level set, i.e. $L_V := \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$. Hence, for $\sigma \in \mathbb{R}_{\geq 0}$, $L_{V(\sigma)} := \{x \in \mathbb{R}^n \mid V(x) \leq \sigma\}$. A square matrix $W \in \mathbb{R}^{n \times n}$ is an $M$-matrix if $W_{i,j} \geq 0 \forall i \neq j$.

### B. Technical background

Let us consider nonlinear control-affine systems

$$\dot{x} = f(x) + g(x)u,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally-bounded functions. We also consider the following notion of control Lyapunov function.

**Definition 1 (CLF).** A positive definite, radially unbounded, smooth away from zero, function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a control Lyapunov function for [1] if there exists a locally-bounded control law $u : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ we have

$$\nabla V(x)(f(x) + g(x)u(x)) < 0.$$

The following definition is fundamental in the sequel.

**Definition 2 (Control-Sharing Property).** Two control Lyapunov functions $V_1$ and $V_2$ for [1] have the control-sharing property if there exists a locally-bounded control law $u : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ we have the following inequalities simultaneously satisfied:

$$\nabla V_1(x)(f(x) + g(x)u(x)) < 0$$

$$\nabla V_2(x)(f(x) + g(x)u(x)) < 0$$

$V_1$ and $V_2$ have the control-sharing property under constraints $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$ if [3] holds for all $x \in X$ with a constrained control law $u : X \to \mathbb{U}$.

For the class of control-affine differential inclusions

$$\dot{x} \in F(x) + G(x)u,$$

where $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^{n \times m}$ are compact-valued mappings, the previous definitions hold unchanged provided that conditions (2) and (3) holds with $\dot{x} = \varphi + \Gamma u$, for all $(\varphi, \Gamma) \in (F(x), G(x))$.

### II. CONTROL-SHARING PROPERTY FOR LINEAR SYSTEMS

Let us also consider linear time-invariant (LTI) systems

$$\dot{x} = Ax + Bu,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

For second-order systems, we have the following result on the control-sharing property.

**Theorem 1.** Two convex CLFs for [5] do necessarily have the control-sharing property if $n \leq 2$.

**Proof:** We have to show that given $\kappa_1, \kappa_2 : \mathbb{R}^2 \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^2$ we have $\nabla V_i(x)(Ax + B\kappa_i(x)) < 0$, for $i = 1, 2$, then for all $x \in \mathbb{R}^2$ there exists $u \in \mathbb{R}^m$ such that the two inequalities $\nabla V_1(x)(Ax + Bu) < 0$ and $\nabla V_2(x)(Ax + Bu) < 0$ can be simultaneously satisfied.

Without any restriction, we assume $m = 1$, so that $B \in \mathbb{R}^{2 \times 1}$, otherwise the proof would be trivial. Assume by contradiction that $V_1$ and $V_2$ do not share a common control, i.e. there exists a point $z \neq 0$ such that the two inequalities (5a, 5b) are not simultaneously satisfied.

If $z$ and $B$ are aligned, namely $z = \lambda B$ for some $\lambda \neq 0$, we can take $u = -c/\lambda$, for some $c > 0$, so that we get

$$\nabla V_1(z)(Az + Bu) = \nabla V_1(z)Az - c \nabla V_1(z)z < 0 \quad (6)$$

$$\nabla V_2(z)(Az + Bu) = \nabla V_2(z)Az - c \nabla V_2(z)z < 0. \quad (7)$$

Since $V_1$ and $V_2$ are convex and positive definite, we have $\nabla V_1(z)z > 0$ and $\nabla V_2(z)z > 0$, therefore for $c$ large enough we have (6) and (7) simultaneously satisfied.

Let $z$ and $B$ be not aligned and hence consider the state transformation $\hat{z} := [Bz]^{-1}z$, so that $\hat{B} := [Bz]^{-1}B = (1, 0)^\top$ and $\hat{z} := [Bz]^{-1}z = (0, 1)^\top$ as in Figure [1]. We make this transformation for ease of understanding, so that in the sequel we consider $z = (0, 1)^\top$ and $B = (1, 0)^\top$.

Then consider the equation $\dot{z} = (Az + Bu) = -\omega z$ in the unknown $u$ and $\omega$, or equivalently $[Bz]^{(-1)}z = -Az$, which has unique solution as $[Bz] = I_2$. Multiplying both sides by...
We finally get a contradiction because \( z \) is in the cone generated by \( v \) and \( y \), therefore \( z = \alpha v + \beta y \) for some \( \alpha, \beta > 0 \), and \( z^\top A z = \alpha z^\top A v + \beta z^\top A y < 0 \), contradicting the fact that \( z^\top A z \geq 0 \).

However, even for second-order systems, the previous result is not “robust”. Consider the class of Linear Differential Inclusions (LDIs)

\[
\dot{x} \in \text{co} \{A_i x + B_i u \mid i \in [1, N]\},
\]

for some \( N > 0 \), \( A_i \in \mathbb{R}^{n \times n} \) and \( B_i \in \mathbb{R}^{n \times m} \) for all \( i \in [1, N] \). The result of Theorem 1 does not hold for this class of systems according to the following result.

**Proposition 2.** Two CLFs for (8) do not necessarily have the control-sharing property.

In general, for \( n > 2 \), the control-sharing property does not hold even for LTI systems.

**Proposition 3.** Two CLFs for (8) do not necessarily have the control-sharing property if \( n > 2 \).

## III. Gradient-type merging control Lyapunov functions

We start by considering as an example the simple double integrator system

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

with constraints \( ||x||_\infty \leq 1 \), \( ||u||_\infty \leq 1 \).

A typical problem is to choose between a CLF \( V_1(x) \) assuring a “large” domain of attraction, see Figure 2 (top), or a function which is “locally optimal” in some sense, such as \( V_2(x) = x^\top P x \). The main idea is compromising the two given functions by a non-homogeneous one which looks like \( V_2(x) \) close to 0 and like \( V_1(x) \) far from 0 as in Fig. 2 (bottom). A CLF with such characteristics is an example of what we call (regular) gradient-type merging CLF.

### A. Merging homogeneous CLFs

In the sequel, we will mainly be concerned with positively homogeneous CLFs.

**Standing Assumption 1.** Functions \( V_1, V_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) are two positively homogeneous CLFs of order 2, respectively in \( \mathcal{L}_{V_1} \) and \( \mathcal{L}_{V_2} \).

Assuming homogeneous CLFs is a limitation. Choosing a degree 2 is without loss of generality because, if \( \dot{V}(x) \leq -\eta V(x) \), then \( V(x) \leq V(p) \leq -\eta p V(p) \) for any \( p > 0 \).

**Definition 3** (Gradient-type merging CLF). Let \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) be positive definite and smooth away from zero. \( V \) is a gradient-type merging candidate if there exist two continuous functions \( \gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) such that \( (\gamma_1(x), \gamma_2(x)) \neq (0, 0) \) and

\[
\nabla V(x) = \gamma_1(x) \nabla V_1(x) + \gamma_2(x) \nabla V_2(x).
\]

\( V \) is a gradient-type merging CLF if, in addition, it is a CLF.
Definition 5. A control law \( u : \mathbb{R}^n \to \mathbb{R}^m \) is regular if it is continuous and for any \( x \in \mathbb{R}^n \) we have
\[
\lim_{\lambda \to 0^+} \frac{u(\lambda x)}{\lambda} = \bar{u}_x, \text{ with } ||\bar{u}_x|| < \infty.
\]

For instance, for linear controllers \( u(x) = Kx \), we just have \( \bar{u}_x = Kx \). Definition 5 basically is a “small control property”, which means that \( u(x) \) goes to 0 at least linearly as \( x \to 0 \).

For linear systems (5), we have the following result for the regular gradient-type merging.

Theorem 5. The following statements are equivalent for (5).

1) There exists a regular gradient-type merging CLF associated with a regular control.
2) Any gradient-type merging is a CLF associated with a regular control.
3) \( V_1 \) and \( V_2 \) share a regular control.

We can relate our “merging” CLFs to the literature on “blending” CLFs [20] and “uniting” CLFs [17, 19] as follows. In [20, Theorem 9.1], it is shown that from the knowledge of two CLFs \( V_1, V_2 \), it is possible to build up a “blending” CLF of the form \( V(x) = \min\{V_1(x), \epsilon V_2(x) + d\} \), for appropriate \( c, \epsilon \geq 0 \), so that \( V \) necessarily admits a stabilizing controller \( \kappa : \mathbb{R}^n \to \mathbb{R}^m \) of the form \( \kappa(x) \in \{\kappa_1(x), \kappa_2(x)\} \). We show that even for linear systems (5), the result does not necessarily hold for gradient-type merging CLFs, namely because of the differentiability property of gradient-type merging candidates.

Proposition 6. Assume \( \kappa_1, \kappa_2 : \mathbb{R}^n \to \mathbb{R}^m \) are control laws respectively associated with \( V_1 \) and \( V_2 \). Then, even for linear systems (5), a regular gradient-type merging CLF \( V \) does not necessarily admit a control law of the kind \( \kappa(x) \in \{\kappa_1(x), \kappa_2(x)\} \).

Remark 2. For nonlinear control-affine systems, [19, Section 2.2] shows that there exists a topological obstruction in uniting a local and a global controller by means of a static time-invariant continuous control law. It follows from the proof of Proposition 6 see Appendix B-C that such an obstruction is also valid for the class of linear systems whenever we look for a controller of the kind used in the proof of [20, Theorem 9.1].

B. Gradient-type merging for differential inclusions

We now consider nonlinear differential inclusions (4) and we provide the following results.

Theorem 7. If \( V_1 \) and \( V_2 \) have the control-sharing property for (4), then any gradient-type merging is a CLF.

Theorem 8. Assume that, in (4), the mapping \( G \) is single-valued. Then the following statements are equivalent for (4).

1) Any gradient-type merging is a CLF.
2) \( V_1 \) and \( V_2 \) have the control-sharing property.

The result of Theorem 8 does also apply to LDIs (8) having \( B_i = B \) for all \( i \in [1, N] \).
IV. CONDITIONS FOR THE EXISTENCE OF A COMMON CONTROLLER

In this section we consider the class of LDIs [8] and we propose several matrix inequality conditions for the existence of a common controller between the CLFs $V_1$ and $V_2$. For ease of presentation, the matrix conditions presented next do not include the control constraints; however, it is worth mentioning that they can be considered without conceptual difficulties. We address the following classes of homogeneous functions: (symmetric) polyhedral, quadratic, max of quadratics and truncated ellipsoid.

Remark 3. Note that some of the mentioned functions are non-smooth. However, we can apply the smoothing procedure in [25]. For instance, if $\|Fx\|_2^2$ is a polyhedral CLF (PCLF) with a certain control law $\kappa$ for an LDI [8], the same control law $\kappa$ assures that $\|Fx\|_p^2$ is a Lyapunov function if $p > 0$ is taken large enough [25]. Therefore if the CLF $V_1(x) = \|Fx\|_\infty^2$ shares a control with the CLF $V_2$, then also $\|Fx\|_2^2$ does for $p$ sufficiently large.

Let $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a positive definite polyhedral function and let $X = [x_1 \mid x_2 \mid ... \mid x_s] \in \mathbb{R}^{n \times s}$ be the matrix whose columns are the vertices of $\mathcal{L}_{V_p}$, i.e.

$$V_p(x) := \min \left\{ \sum_{j=1}^s \alpha_j \bigg| \sum_{j=1}^s \alpha_j x_j = x \right\}. \quad (10)$$

Then $V_p$ is a PCLF for (8) and only if there exist $\eta > 0$, $\mathcal{M}$-matrices $W_1, W_2, ..., W_N \in \mathbb{R}^{s \times s}$ and $U \in \mathbb{R}^{m \times s}$ such that for all $i \in [1,N]$ we have

$$A_i X + B_i U = X W_i, \quad T_i W_i \leq -\eta T_i^T. \quad (11)$$

The following result is technical, and it will be exploited later. It states that given a PCLF $V_p$, represented by a matrix $X$, we can always add a “redundant vertex”, either in the interior int$\mathcal{L}_{V_p}$ or on the boundary $\partial \mathcal{L}_{V_p}$, achieving a feasibility condition similar to (11).

Lemma 9. Assume (11) is feasible. Given $\alpha_1, \alpha_2, ..., \alpha_s \geq 0$ such that $\sum_{j=1}^s \alpha_j = 1$, consider $\bar{x} = \sum_{j=1}^s \alpha_j x_j$ and let $\bar{X} := [X \mid \bar{x}] \in \mathbb{R}^{n \times s+1}$. Then there exist $\mathcal{M}$-matrices $\bar{W}_1, \bar{W}_2, ..., \bar{W}_N \in \mathbb{R}^{s \times s}$ such that for all $i \in [1,N]$ we have

$$\bar{A}_i \bar{X} + \bar{B}_i U = \bar{X} \bar{W}_i, \quad T_i \bar{W}_i \leq -\eta T_i^T. \quad (12)$$

where $\bar{U} := [U \mid \bar{u}]$ with $U = [u_1 \mid u_2 \mid ... \mid u_s]$ and $\bar{u} = \sum_{j=1}^s \alpha_j u_j$.

Let us consider the case of two PCLFs. In view of Lemma 9 according to the construction of Figure 3, for any vertex $x_k$ of $\mathcal{L}_{V_1}$ we add a “fictitious” redundant vertex $\tilde{x}_k$ on the boundary of $\mathcal{L}_{V_2}$ aligned with $x_k$ and vice-versa, so augmenting both the describing matrices $X_1$ and $X_2$. We have the following result.

Theorem 10. Assume that $V_1$ and $V_2$ are two PCLFs of the form (10), with $X_1 = [x_1^1 \mid ... \mid x_k^1]$ and $X_2 = [x_1^2 \mid ... \mid x_k^2]$, respectively. For each column of $X_1$, namely each vertex $x_k$, take point $\tilde{x}_k := cx_k^1 \in \partial \mathcal{L}_{V_2}$, for some $c > 0$ (see Fig. 3). Analogously, take $x_k := c \tilde{x}_k \in \partial \mathcal{L}_{V_1}$, for some $c > 0$. Define $\bar{x}_k^1 := [x_k^1 \mid x_k^2]$ for $\mathbb{R}^{n \times (s_1+s_2)}$ and $\bar{x}_k := [\bar{x}_k^1 \mid \bar{x}_k^2] \in \mathbb{R}^{n \times (s_1+s_2)}$.

Then $V_1$ and $V_2$ have the control-sharing property if and only if there exist $\eta > 0$, $\mathcal{M}$-matrices $W_1^1, ..., W_N^1 \in \mathbb{R}^{(s_1+s_2) \times (s_1+s_2)}$, $W_1^2, ..., W_N^2 \in \mathbb{R}^{(s_1+s_2) \times (s_1+s_2)}$, and $U \in \mathbb{R}^{m \times (s_1+s_2)}$ such that for all $i \in [1,N]$ we have

$$A_i \bar{x}_k^1 + B_i U = \bar{x}_k^1 W_i^1, \quad T_i W_i^1 \leq -\eta T_i^T \quad (13a)$$
$$A_i \bar{x}_k^2 + B_i U = \bar{x}_k^2 W_i^2, \quad T_i W_i^2 \leq -\eta T_i^T \quad (13b)$$

simultaneously satisfied.

We now consider the control-sharing between polyhedral and quadratic CLF (QCLF) for (8).

Theorem 11. Assume that $V_1 = V_p$ as in (10) and $V_2(x) = x^T P x$ respectively are PCLF and QCLF for (8). Let $r$ be the number of facets of $\mathcal{L}_{V_1}$ and let $V_k$ be the set of the vertices belonging to the $k$th facet, whose cardinality is $s_k \in [1,s]$. For all $k \in [1,r]$ and $i \in [1,N]$, define the matrices $S_{k,i}(\eta,U) \in \mathbb{R}^{s_k \times s_k}$ componentwise as

$$[S_{k,i}(\eta,U)]_{j,j} := x_k^j P ((A_i + \eta I) x_j + B_i u_j) + x_j^T P ((A_i + \eta I) x_k + B_i u_k), \quad (14)$$

where $x_h, x_j \in V_k$. Then $V_1$ and $V_2$ have the control-sharing property if there exist $\eta > 0$, $\mathcal{M}$-matrices $W_1, W_2, ..., W_N \in \mathbb{R}^{s \times s}$ and $U = [u_1 \mid ... \mid u_s] \in \mathbb{R}^{m \times s}$ such that (11) holds and the matrices $-S_{k,i}(\eta,U)$ are copositive for all $k \in [1,r]$ and $i \in [1,N]$.

The condition proposed in Theorem 11 requires the solution of a copositive programming problem. This problem is convex, but still hard to solve. A sufficient condition which can be checked via LP is that the matrices $S_{k,i}(\eta,U)$ have non-positive elements.

1. $M$ is copositive if $x^T M x \geq 0$ for all nonnegative vectors $x$. 
Corollary 12. Under the assumptions of Theorem \[23\] \(V_1\) and \(V_2\) have the control-sharing property if there exist \(\eta > 0\), matrices \(W_1, W_2, \ldots, W_N \in \mathbb{R}^{n \times n}\) and \(U \in \mathbb{R}^{m \times n}\) such that (11) holds and the elements (14) of \(S_{k,i}(\eta, U)\) are non-positive for all \(k \in [1, r]\) and \(i \in [1, N]\).

Then, we consider positive definite 0-symmetric functions \(V_s : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) defined as

\[
V_s(x) := \max \{ x^T Q_k x \mid k \in [1, s]\}
\]

for some \(Q_1, Q_2, \ldots, Q_s \succeq 0\), hence covering the case of symmetric polyhedral functions, truncated ellipsoids and max of quadratics.

Theorem 13. Assume that \(V_1 = V_s (15)\) and \(V_2(x) = x^T P x\) respectively are CLF and QCLF for \(\mathcal{L}\). Then \(V_1\) and \(V_2\) have the control-sharing property if there exist \(\eta > 0\), \(\lambda_{i,j,k} \geq 0\), \(K_k \in \mathbb{R}^{n \times n}\), for \(i = 1, 2, \ldots, N\), and \(j, k = 1, 2, \ldots, s\), such that

\[
(A_i + B_i K_k)^T Q_k + Q_k (A_i + B_i K_k) \leq -2\eta Q_k + \sum_{i=1}^s \lambda_{i,j,k} (Q_j - Q_k)
\]

for all \(i \in [1, N]\), \(k \in [1, s]\).

Remark 5. Theorem 7 is more general than [23] Theorem 2], because condition (9) relies on a piecewise-linear common controller, rather than a linear common controller as in [23] matrix conditions (11).}

V. The R-composition as a Gradient-Type Merging

In this section, we investigate the “R-composition” between two homogeneous CLFs proposed in [23], which is shown to be a regular merging-type merging CLF in the sequel. The composition consists of the following steps.

1) Define \(R_1, R_2 : \mathbb{R}^n \to \mathbb{R}\) as \(R_i(x) = 1 - V_i(x), i = 1, 2\).

2) For fixed \(\phi > 0\), define \(R_\lambda : \mathbb{R}^n \to \mathbb{R}\) as

\[
R_\lambda(x) := \rho(\phi) \left( \phi R_1(x) + R_2(x) - \sqrt{\phi^2 R_1(x)^2 + R_2(x)^2} \right),
\]

where \(\rho(\phi) := \left( \phi + 1 - \sqrt{\phi^2 + 1} \right)^{-1}\) is the normalization factor [23] Section 2].

3) Define the “R-composition” \(V_\lambda : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) as

\[
V_\lambda(x) := 1 - R_\lambda(x).
\]

It turns out that [23] Proof of Theorem 1]

\[
\nabla V_\lambda(x) = \rho(\phi) \left[ \phi c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x) \right],
\]

where \(c_1, c_2 : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) are defined as

\[
c_1(\phi, x) := 1 + \frac{-\phi R_1(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}},
\]

\[
c_2(\phi, x) := 1 + \frac{-R_2(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}}.
\]

It follows from the properties of the “R-functions”, see Appendix E that \(V_\lambda\) is positive definite (Lemma [17], differen-
tiable in int\(\mathcal{L}_V\), (Lemma [18]), and that \(\mathcal{L}_V = \mathcal{L}_V \cap \mathcal{L}_V\) (Lemma [19]).

The function \(V_\lambda\), namely the merging of \(V_1\) and \(V_2\) from Standing Assumption [1] will be used as a candidate CLF later on.

Proposition 14. \(V_\lambda\) is a gradient-type merging candidate.

We can now show that \(V_\lambda\) is a regular merging-type candidate with arbitrarily small tolerance.

Proposition 15. Let \(\mathcal{L}_V \supset \mathcal{L}_{V_1}\). Then for any \(\varepsilon > 0\) and \(\delta \in (0, 1)\) there exists \(\phi > 0\) such that for all \(\phi \geq \delta\) we have that \(V_\lambda\), with domain \(\mathcal{L}_{(V_1, \varepsilon)}\), is a regular gradient-type merging candidate with tolerance \(\varepsilon\).

According to Theorem [4] and Theorem [7] if \(V_1\) and \(V_2\) are CLFs for \(\mathcal{L}\) and share a constrained controller \(\kappa\), then \(\kappa\) is admissible as well for \(V_\lambda\), which turns out to be a CLF for \(\mathcal{L}\) under constraints. In this case, we will refer to the CLF \(V_\lambda\) as Control Lyapunov R-Function (CLRF).

It follows from the proof of Lemma [17] that, independently from \(\phi > 0\), the unit level set of \(V_\lambda\) is \(\mathcal{L}_{V_\lambda} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2}\) and \(\mathcal{L}_{V_\lambda} \supset \mathcal{L}_{V_1}\), \(\mathcal{L}_{V_\lambda} \supset \mathcal{L}_{V_2}\), \(\mathcal{L}_{V_\lambda} \supset \mathcal{L}_{V_1}\) point-wise in int\(\mathcal{L}_{V_\lambda}\). Moreover, according to Lemmas [20, 21, 22], we have \(\nabla V_\lambda(x) \xrightarrow{\phi \to \infty} \nabla V_2(x)\) and \(\nabla V_\lambda(x) \xrightarrow{\phi \to 0^+} \nabla V_1(x)\), point-wise in int\(\mathcal{L}_{V_\lambda}\).

This particular property of fixing the “external” shape, while making the “inner” one “close” to any given choice can be exploited to fix a “large” DoA while achieving “locally-optimal” closed-loop performances.

Remark 5. We remind that the (smoothed) polyhedral functions of the kind [27, 23, 29, 25], composite quadratics [24] and the convex hull of quadratics [7] are universal classes of homogeneous functions for the stability/stabilizability of LDIs [8]. Exploiting Lemma [22] we can merge one of them with any \(V_2\) (homogeneous of degree 2) to indeed achieve a new class of universal non-homogeneous Lyapunov functions.

A. Controller design under constraints

We now investigate the existence of a continuous locally-optimal control under constraints \(x \in \mathcal{L}_{V_1}\) and \(u \in U \subseteq \mathbb{R}^m\) which is closed (possibly compact) and convex. For simplicity, we consider [8] with \(B_i = B\) for all \(i \in [1, N]\). Since the CLF \(V_\lambda\) is differentiable, in principle, the existence of a stabilizing control law \(\kappa\) continuous with the exception of the origin, or
including $x = 0$ if $V_\lambda$ satisfies the small control property\(^4\) could be proved by using the arguments in \cite[Chapters 2–4]{31}.

To have $\mathcal{L}_{V_\lambda} = \mathcal{L}_{V_1}$, we preliminary scale $V_2$ so that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. In light of Theorem \cite{7} we formulate the control-sharing assumption, which can be checked using the results in Section IV.

**Assumption 1.** Functions $V_1$ and $V_2$ have the control-sharing property under constraints $x \in \mathcal{L}_{V_1}$, $u \in \mathcal{U}$. Associated with $V_2$ there is an “optimal” continuous control law $\kappa_2 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\kappa_2(x) \in U$ for all $x$ in a neighborhood of the origin.

It follows from the proof of Theorem \cite{7} that, under Assumption \cite{1} $V_\lambda$ is a CLRF for (7) under constraints. Namely, since for all $x \in \mathbb{R}^n$ we have $\min \{V_1(x), V_2(x)\} \leq V_\lambda(x) \leq \max \{V_1(x), V_2(x)\}$ \cite[Proposition 1, Section 4.2]{23}, there exists $\eta > 0$ such that the following convex-valued mapping of admissible (constrained) controls is non-empty for all $x \in \mathcal{L}_{V_\lambda}$.

$$U(x) := \left\{ u \in \mathcal{U} \mid \max_{v \in [1, N]} \nabla V_\lambda(x)(A_1 x + B u) + \eta x^\top x \leq 0 \right\}.$$ \hfill (21)

We indeed propose the control law

$$\kappa(x) := \arg \min \{ \|v - \kappa_2(x)\| \mid v \in U(x) \}.$$ \hfill (22)

**Theorem 16.** Suppose Assumption \cite{7} holds. Then the control law $\kappa$ \cite{22} associated with $V_\lambda$ \cite{18} is continuous, satisfies the constraints in $\mathcal{L}_{V_1}$, and is locally optimal.

**Remark 6.** In the case of constrained “linear-quadratic” (LQ) stabilization, the approximate Hamilton–Jacobi–Bellman control

$$\tilde{\kappa}(x) := \arg \min_{v \in U(x)} \{ \nabla V_\lambda(x)(A x + B v) + x^\top Q x + v^\top R v \}$$

has been proposed in [23 Section 5]. An advantage of $\kappa$ \cite{22} over $\tilde{\kappa}$ is that, according to Theorem 16, local optimality is here guaranteed.

**B. Illustrative example**

We address the constrained stabilization of a simplified inverted pendulum, whose dynamics is given by the nonlinear differential equation $I \dot{\theta}(t) = mg l \sin(\theta(t)) + \tau(t)$. The goal is the stabilization of $(\theta, \dot{\theta})$ to the origin, under the constraints $|\theta| \leq \frac{\pi}{4}$, $|\dot{\theta}| \leq \frac{\pi}{4}$ and $|\tau| \leq 2$. With notation $x_1 = \theta$, $x_2 = \dot{\theta} = \dot{x}_1$, $u = \tau$ and $w(x) := \left\{ \frac{\sin(x_1)}{x_1} \mid |x_1| \leq \frac{\pi}{4} \right\}$, the following constrained uncertain linear model can be derived.

$$\dot{x} \in \begin{bmatrix} 0 & 1 \\ \frac{a w(x)}{b} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u,$$ \hfill (23)

where $a = (mg l / I)$, $b = (1 / l)$; $w(x) \approx [0.89, 1]$, $w(0) = 1$; $|x_1| \leq \pi/4$, $|x_2| \leq \pi/4$, $|u| \leq 2$. The numerical parameters used in the simulation are $I = 0.05$, $m = 0.5$, $g = 9.81$, $l = 0.3$.

\(^4\)A CLF $V$ satisfies the small control property if, for $u := \kappa(x)$, we have that for all $v \in \mathbb{R}^n$ there exists $\epsilon \in \mathbb{R}^n$ so that, whenever $\|x\| < \epsilon$ we have $\|u\| < v$ \cite[30]{30}.

We adopt the infinite-horizon quadratic performance cost

$$J(x, u) := \int_0^\infty (\|x(t)\|^2_Q + \|u(t)\|^2_R) dt,$$

with weight matrices $Q = I_2$, $R = 10$. Let us indeed define the locally-optimal (i.e. for $w \equiv 1$) cost function $V_2(x) = x^\top P x$, where $P$ is the unique solution of the Algebraic Riccati Equation. It can be shown that function $V_1(x) = \|F x\|_C^2$, with

$$F = \begin{bmatrix} 0 & 1.53 \\ 1/4 \pi & 0.51 \end{bmatrix}^\top,$$

is a PCLF for the constrained LDI \cite{22} and therefore also for the constrained nonlinear system. Then we define the smoothed PCLF $V_1(x) = \|F x\|_C^{\phi}$ \cite{25} and we indeed focus on the DoA $\mathcal{L}_{V_1}$. Let us also define $V_2$ scaling $V_2$, so that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. Since the LMI condition \cite{15} is satisfied under constraints, $V_1$ and $V_2$ share a constrained controller in $\mathcal{L}_{V_1}$, therefore any gradient-type merging is a CLF. We indeed construct a CLRF $V_\lambda$ with $\phi = 20$.

Now, $V_1$ has a “large” DoA but it induces a “poor” performance when used with gradient-based controllers of the kind \cite{22} (Figure 4 in fact shows that the constraint $u \in \mathcal{U}(x)$ \cite{21} with $V_1$ in place of $V_\lambda$ may be “too restrictive”). On the other hand, $V_2$ is locally optimal, but both gradient-based controllers, for instance \cite{22} with $V_2$ in place of $V_\lambda$, and the standard LQ regulator yield constraint violations, even in the case $w \equiv 1$. We notice that $V_\lambda$, see Figure 5 with controller \cite{22}, inherits the benefits of $V_1$ (“large” DoA under constraints) and $V_2$ (local optimality). For the linearized system (i.e. for $w \equiv 1$), our extensive Monte Carlo numerical experiments show that the closed-loop performance is “quite close” to the constrained “global optimal” (obtained via a receding “long”-horizon controller, under a “fine” system discretization).

**VI. Conclusion**

The problem of merging two Lyapunov functions is considered important for several applications, mainly because when concerning constraints, robustness and optimality, a single Lyapunov function is typically suitable for one of these goals, but ineffective for the others.

Previous results show how to combine Lyapunov functions if these share a common control in a suitable region of the state space. For the class of nonlinear control-affine systems,
both differential equations and inclusions, we have shown the equivalence between the control-sharing property and the existence of merging control Lyapunov functions.

In order to guarantee the existence of a common control law, linear programs and linear matrix inequalities conditions have been presented for the class of linear differential inclusions.

As an example of merging procedure, a constructive technique based on the R-composition has been given. Further numerical experiments on practical case studies have to be presented. From our experience, our approach is quite close to the constrained global optimality, but no “close form” bounds have been given.

APPENDIX A
PROOFS OF SECTION III

A. Proof of Proposition 2

Proof: We show a numerical example for \( n = 3 \), in which two QCLFs \( V_1(x) = x^\top P_1 x \) and \( V_2(x) = x^\top P_2 x \) do not share a common controller.

Consider (5) with
\[
A = \begin{bmatrix} -1.990 & -1.135 & -1.063 \\ -1.745 & 0.536 & -0.429 \end{bmatrix}, \quad B = \begin{bmatrix} -1.925 \\ -0.794 \\ -1.243 \end{bmatrix}.
\]

Note that the eigenvalues of \( A \) are \( \{0.276, -1.772 \pm i0.114\} \).

Let us consider
\[
P_1 = \begin{bmatrix} 35.3372 & 27.5998 & -39.0922 \\ 27.5998 & 21.4164 & -30.4326 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} 0.00031 & 0.04321 & -0.01465 \\ 0.04321 & 80.5659 & -39.5654 \end{bmatrix}.
\]

With the linear controllers \( \kappa_1(x) = K_1 x \) and \( \kappa_2(x) = K_2 x \), being \( K_1 = (0.5037, 0.5799, -0.2013) \) and \( K_2 = (4.5451, 4.5697, -0.0669) \), we have \( (A+BK_1)\top P_1 + P_1 (A+BK_1) \not< -\epsilon_1 I_n \), for \( i = 1, 2 \), with \( \epsilon_1, \epsilon_2 \geq 10^{-4} \). Therefore \( x^\top P_1 x \) and \( x^\top P_2 x \) are CLFs.

Then, we show that for the state \( \bar{x} = (-0.329, -1.094, -1.537)\top \), there exists a common control \( u \in \mathbb{R} \), i.e. the following system of equations is not admissible.
\[
\begin{align*}
\nabla V_1(\bar{x})(A_1 \bar{x} + Bu) &< 0 \\
\nabla V_2(\bar{x})(A_2 \bar{x} + Bu) &< 0
\end{align*}
\]
\[
\nabla V_2(\bar{x})(A_2 \bar{x} + Bu) < 0
\]

In fact, we have \( \frac{1}{2} \nabla V_1(\bar{x})A_2 \bar{x} = \bar{x}^\top P_1 A_2 \bar{x} = 6.94 \), \( \frac{1}{2} \nabla V_1(\bar{x})B = \bar{x}^\top P_1 B = 11.74 \), therefore \( u < -0.59 < 0 \); however \( \frac{1}{2} \nabla V_2(\bar{x})A_1 \bar{x} = \bar{x}^\top P_2 A_1 \bar{x} = 1.89 \) and \( \frac{1}{2} \nabla V_2(\bar{x})B = \bar{x}^\top P_2 B = -1.48 \), therefore \( u > 1.28 > 0 \).

B. Proof of Proposition 3

Proof: We show a numerical example for \( n = 3 \), in which two QCLFs \( V_1(x) = x^\top P_1 x \) and \( V_2(x) = x^\top P_2 x \) do not share a common controller.

Consider (5) with
\[
A = \begin{bmatrix} -1.990 & -1.135 & -1.063 \\ -1.745 & 0.536 & -0.429 \end{bmatrix}, \quad B = \begin{bmatrix} -1.925 \\ -0.794 \\ -1.243 \end{bmatrix}.
\]

That clearly is not feasible.

Remark 7. The sets of equations (24) and (25) are not influenced by any scaling of the matrices \( P_i \), meaning that the set of admissible solutions remains the same for \( P_i \mapsto \delta_i P_i \), \( \delta_i > 0 \), \( i = 1, 2 \). Such a scaling would influence \( \epsilon_1, \epsilon_2 \) in \( (A_j + BK_i)^\top P_i + P_i (A_j + BK_i) \not< -\epsilon_i I_n \) in the following sense. For any \( \bar{\epsilon}_1, \bar{\epsilon}_2 > 0 \), there exist \( \delta_1, \delta_2 > 0 \) such that \( (A_j + BK_i)^\top \delta_i P_i + \delta_i P_i (A_j + BK_i) \not< -\epsilon_i I_n \) for \( i = 1, 2 \).

That is to say that we cannot run into numerical problems caused by “too small” \( \epsilon_1, \epsilon_2 \).

APPENDIX B
PROOFS OF SECTION III

A. Proof of Theorem 4

Proof: \( V \) is a CLF if and only if for any \( x \in \mathbb{R}^n \) there exists \( u \in \mathbb{R}^m \) such that \( \nabla V(x)(f(x) + g(x)u) < 0 \). Assume that \( V \) is a CLF and let \( x \) be fixed. By definition, for any \( \gamma_1, \gamma_2 \geq 0 \) with \( (\gamma_1, \gamma_2) \neq (0, 0) \), there exists \( u \in \mathbb{R}^m \) such
that $(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x))(f(x) + g(x)u) < 0$, or equivalently for any $(a_1, a_2) \in \mathcal{A} := \{(a, b) \in [\mathbb{R}_{\geq 0}]^2 \mid a + b = 1\}$ there exists $u \in \mathbb{R}^m$ such that
\[(a_1 \nabla V_1(x) + a_2 \nabla V_2(x))(f(x) + g(x)u) < 0.
\]
Therefore we have
\[\max_{(a_1, a_2) \in \mathcal{A}} \inf_{u \in \mathbb{R}^m} (a_1 \nabla V_1(x) + a_2 \nabla V_2(x))(f(x) + g(x)u) < 0.
\]

Since $\mathcal{A}$ is compact and $\mathbb{R}^m$ is closed, and the function in (26) is linear in both $(a_1, a_2)$ and $u$, we can exchange “max” and “min” \cite{1} Corollary 37.3.2] to get the following equivalent condition.
\[\max_{(a_1, a_2) \in \mathcal{A}} \inf_{u \in \mathbb{R}^m} (a_1 \nabla V_1(x) + a_2 \nabla V_2(x))(f(x) + g(x)u) = \inf_{u \in \mathbb{R}^m} \max_{(a_1, a_2) \in \mathcal{A}} (a_1 \nabla V_1(x)(f(x) + g(x)u) + a_2 \nabla V_2(x)(f(x) + g(x)u) < 0 \Longleftrightarrow \inf_{u \in \mathbb{R}^m} \max_{(a_1, a_2) \in \mathcal{A}} \{a_1 \nabla V_1(x)(f(x) + g(x)u) + a_2 \nabla V_2(x)(f(x) + g(x)u)\} < 0.
\]

The last inequality is equivalent to the existence of a common controller. The result follows as all the considered inequations are equivalent. \hfill \blacksquare

B. Proof of Theorem \ref{thm:main}

Proof: In view of Theorem \ref{thm:main} we need only to prove that if there exists a regular gradient-type merging CLF, then the two functions have the control-sharing property.

Therefore, by assumption we have $(\gamma_1(x) \nabla V_1(x) + \gamma_2(x) \nabla V_2(x))(Ax + Bu(x)) < 0$ for some locally bounded $u(x)$.

Given a unit vector $v \in \mathbb{R}^m, ||v|| = 1$, consider the ray $\mathcal{R} := \{\lambda v \in \mathbb{R}^m \mid \lambda > 0\}$. Since the functions are homogeneous, their gradients along $\mathcal{R}$ are aligned, namely for all $x = \lambda v$ we have $\nabla V_1(x) = \lambda \nabla V_1(v)$ and $\nabla V_2(x) = \lambda \nabla V_2(v)$ for some $p, q > 0$. Therefore we have
\[(\gamma_1(\lambda v)\lambda^p \nabla V_1(v) + \gamma_2(\lambda v)\lambda^q \nabla V_2(v))(\lambda Av + Bu(\lambda v)) < 0,
\]

or equivalently (divide by $\gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q > 0$ and by $\lambda > 0$) to get
\[
\begin{pmatrix}
\frac{\gamma_1(\lambda v)\lambda^p}{\gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q} & \nabla V_1(v) \\
= \alpha_1(\lambda) & \\
\frac{\gamma_2(\lambda v)\lambda^q}{\gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q} & \nabla V_2(v) \\
= \alpha_2(\lambda)
\end{pmatrix}
(Av + B\omega) < 0,
\]

where we define
\[\omega := \frac{u(\lambda v)}{\lambda}.
\]

Denote by $\bar{\lambda}$ the value of $\lambda$ such that $\tilde{\lambda} v \in \delta \mathcal{L}_V$, i.e. $V(\lambda v) = 1$. For all $\lambda \in [0, \bar{\lambda}]$, we have $(\alpha_1(\lambda), \alpha_2(\lambda)) \in \mathcal{A} := \{(a, \beta) \in [\mathbb{R}_{\geq 0}]^2 \mid a + \beta = 1\}$. Moreover as $\lambda$ goes from 0 to $\bar{\lambda}$, both $\alpha_1(\lambda)$ and $\alpha_2(\lambda) = 1 - \alpha_1(\lambda)$ assume all values from 0 to 1. This means that for all $(a_1, a_2) \in \mathcal{A}$ there exists $\omega \in \mathbb{R}^m$ such that $(a_1 \nabla V_1(v) + a_2 \nabla V_2(v))(Av + B\omega) < 0$, i.e.
\[\max_{(a_1, a_2) \in \mathcal{A}} \inf_{u \in \mathbb{R}^m} (a_1 \nabla V_1(v) + a_2 \nabla V_2(v))(Av + B\omega) < 0.
\]

To complete the proof we just need to apply the same min-max argument of the proof of Theorem \ref{thm:main}. \hfill \blacksquare

C. Proof of Proposition \ref{pro:1}

Proof: We prove the claim by means of an example with $n = m = 2$. Consider the linear system $\dot{x} = u$, along with the linear controllers $\kappa_1(x) = K_1 x = \begin{pmatrix} -1 \lambda u \alpha_1(\lambda) & \nabla V_1(v) \\
\lambda v \end{pmatrix} \begin{pmatrix} -\lambda \alpha_1(\lambda) & 1 \end{pmatrix}$, for some $a, \epsilon > 0$. The functions $V_1(x) = \frac{1}{2}(ax_1 + \frac{1}{2}x_2^2), V_2(x) = \frac{1}{2}(\frac{1}{2}x_1^2 + ax_2^2),$ are two QCLFs, respectively with controllers $\kappa_1$ and $\kappa_2$. In fact, since $\nabla V_1(x) = (ax_1, \frac{1}{2}x_2), \nabla V_2(x) = (\frac{1}{2}x_1, ax_2)$, we have $\nabla V_1(x)(Ax + Bu) = -V_1(x)$, for $i = 1, 2$.

Take any gradient-type merging candidate $V(x) = (\gamma_1(x) \nabla V_1(x) + \gamma_2(x) \nabla V_2(x))$ and
\[\{\gamma_1(x) = 1, \gamma_2(x) = 0\} \text{ “far” from the state-space origin and, vice-versa, } \{\gamma_1(x) = 0, \gamma_2(x) = 1\} \text{ “close” to the origin.}
\]

Therefore $V$ is such that $\nabla V(x) = \nabla V_1(x)$ “far” from the origin and $\nabla V(x) = \nabla V_2(x)$ “close” to the origin. The controller $\kappa(x) = -ex$ assures that $V$ a CLF, as $\nabla V(x)(Ax + Bu) = -\epsilon \nabla V(x)x = -\epsilon(\nabla V_1(x) + \nabla V_2(x))x = -\epsilon(a + \frac{1}{a})(x_1^2 + x_2^2)$ is negative definite $\forall \epsilon, a > 0$.

Note that for $a \gg 1$ the vector $\nabla V_1$ is almost “horizontal”, while the vector $\nabla V_2$ is almost “vertical”. Consider the ray (bisector) $\mathcal{R} := \{x = (\xi, \xi), \xi \geq 0\}$. Since $\nabla V$ is continuous, there exists a point $\mathcal{R}$ on the bisector in which $\nabla V$ is aligned to the bisector itself, i.e. there exist $\lambda, \xi \geq 0$ such that $\nabla V(\xi) = -\lambda(\xi, \xi)$. In such a point with both $\kappa_1(\xi)$ and $\kappa_2(\xi)$, we have $\nabla V(\xi)(Ax + Bu(\xi)) = -\lambda(2\epsilon + (\frac{1}{2} - a))\xi^2$ that is strictly positive for $\epsilon \ll 1, a \gg 1$.

D. Proof of Theorem \ref{thm:2}

Proof: By assumption, for all $x \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that the inequalities
\[\max_{(\varphi, \Gamma) \in (F(x), G(x))} \nabla V_1(x)(\varphi + GU) < 0 \iff \max_{(\varphi, \Gamma) \in (F(x), G(x))} \gamma_1 \nabla V_1(x)(\varphi + GU) < 0 \quad (28a)
\]
\[\max_{(\varphi, \Gamma) \in (F(x), G(x))} \nabla V_2(x)(\varphi + GU) < 0 \iff \max_{(\varphi, \Gamma) \in (F(x), G(x))} \gamma_2 \nabla V_2(x)(\varphi + GU) < 0 \quad (28b)
\]
holds simultaneously for any $\gamma_1, \gamma_2 \geq 0$, therefore also the following sum is negative:
\[\max_{(\varphi, \Gamma) \in (F(x), G(x))} \{\gamma_1 \nabla V_1(x)(\varphi + GU)\} + \max_{(\varphi, \Gamma) \in (F(x), G(x))} \{\gamma_2 \nabla V_2(x)(\varphi + GU)\} < 0,
\]
which immediately implies
\[
\max_{(\varphi,\Gamma) \in (F(x),G(x))} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) (\varphi + \Gamma u) < 0.
\]

Having the left-hand side strictly less than zero is equivalent to claim that the gradient-type merging is a CLF. The proof is complete if we notice that \(\gamma_1\) and \(\gamma_2\) have been chosen arbitrarily.

\[\text{\(\blacksquare\)}\]

E. Proof of Theorem 8

Proof: The implication (2) \(\Rightarrow\) (1) follows from Theorem 7. To prove the claim (1) \(\Rightarrow\) (2) we write \(G(x) = g(x)\) to mean that \(G\) is single-valued. Fix arbitrary \(\gamma_1, \gamma_2 > 0\) and define
\[
\tilde{f}(x) := \arg \max_{\varphi \in F(x)} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \varphi.
\]

Now, by assumption we have that
\[
\max_{\varphi \in F(x)} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) (\varphi + g(x)u) < 0,
\]

namely
\[
\max_{\varphi \in F(x)} \left\{ (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \varphi \right\} +
(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) g(x)u < 0.
\]

According to the definition of \(\tilde{f}\), the first term can be written as \((\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \tilde{f}(x)\). Finally, we can just follow the proof of Theorem 4 for the nonlinear system \(\dot{x} = \tilde{f}(x) + g(x)u\).

\[\text{\(\blacksquare\)}\]

APPENDIX C

PROOFS OF SECTION IV

A. Proof of Lemma 9

Proof: For each \(i \in [1, N]\) we explicitly construct \(\tilde{W}_i\) from \(W_i\). Therefore, for ease of notation, let us consider the case of \(N = 1\). The general case easily follows as each \(\tilde{W}_i\) here constructed will depend exclusively on \(W_i\).

Let us denote \(\tilde{W} = [\tilde{w}_1|\tilde{w}_2|...|\tilde{w}_{s+1}]\), where \(\tilde{w}_i \in \mathbb{R}^{s+1}\). Moreover we use the notation \(w_i(p)\) to denote the \(p^{th}\) component of a column vector \(w_i\) and \(w_i([p, q])\) to denote its \(p^{th}\), \((p+1)^{th}\), ..., \((q-1)^{th}\), \(q^{th}\) components.

Then we have
\[
A[x_1|...|x_s|\tilde{x}] + B[w_1|...|w_s|\tilde{u}] = [X|\tilde{x}][w_1|...|w_s|\tilde{w}_{s+1}].
\]

The first \(s\) equations are
\[
A_i x_i + B u_i = [X|\tilde{x}] \tilde{w}_i, \ i \in [1, s],
\]

therefore we can take \(\tilde{w}_i([1, s]) := w_i([1, s])\) and \(\tilde{w}_i(s+1) := 0\). Note that this definition respects the fact that \(\tilde{W}\) has to be an \(\mathcal{M}\)-matrix. The last equation is \(A \tilde{x} = B \tilde{u} = [X|\tilde{x}] \tilde{w}_{s+1}\).

Now the left-hand side can be written as
\[
A \left( \sum_{i=1}^{s} \alpha_i x_i \right) + B \left( \sum_{i=1}^{s} \alpha_i u_i \right) =
X \left( \sum_{i=1}^{s} \alpha_i w_i \right) s+1 (s+1).
\]

while the right-hand side can be written as
\[
X \tilde{w}_{s+1}([1, s]) + \left( \sum_{i=1}^{s} \alpha_i x_i \right) w_{s+1} (s+1) =
X [\tilde{w}_{s+1}([1, s]) + \tilde{w}_{s+1} (s+1) \alpha] (\text{30})
\]

where \(\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_s)^T\).

Therefore, from (29) and (30), it is sufficient to show that, for any given \(\alpha \in (\mathbb{R}_{\geq 0})^s\), there exists \(\tilde{w}_{s+1}\) such that
\[
\sum_{i=1}^{s} \alpha_i w_i = \tilde{w}_{s+1}([1, s]) + \tilde{w}_{s+1} (s+1) \alpha.
\]

For ease of notation, we rename the free variables as \(\xi := \tilde{w}_{s+1}([1, s]) \in (\mathbb{R}_{\geq 0})^s\) and \(\zeta := \tilde{w}_{s+1} (s+1) \in \mathbb{R}\). Then, we define
\[
\tilde{W} := \begin{bmatrix}
w_1 (1) & \cdots & w_s (1) \\
\vdots & \ddots & \vdots \\
w_1 (s) & \cdots & w_s (s)
\end{bmatrix},
\]

Note that, as \(W\) is an \(\mathcal{M}\)-matrix by assumption, we have \(\sum_i w_i (j) \geq 0\) \(\Rightarrow\) \(j \neq i\). Therefore, according to (31) and (32), we have to find \(\xi\) and \(\zeta\) such that \(\tilde{W} \alpha = \xi + \zeta \alpha\), or equivalently:
\[
(\tilde{W} - \zeta I) \alpha = \xi.
\]

Now defining \(\zeta := -\left( \max_{i \in [1, s]} |w_i(i)| \right)\), we get that the matrix \((\tilde{W} - \zeta I)\) has all non-negative entries, therefore \((\tilde{W} - \zeta I)\) becomes a vector of all non-negative components. This means that (33) can be satisfied by choosing \(\xi := (\tilde{W} - \zeta I)\alpha \in (\mathbb{R}_{\geq 0})^s\).

Summarizing, we found an admissible \(\mathcal{M}\)-matrix \(\tilde{W} \in \mathbb{R}^{(s+1) \times (s+1)}\) of the kind
\[
\tilde{W} := \begin{bmatrix}
W & \xi \\
0 & \zeta
\end{bmatrix}.
\]

To conclude the proof, we have to show that \(\tilde{\Gamma}_{s+1}^T \tilde{W} \leq -\eta \tilde{\Gamma}_{s+1}^T\), i.e. that all the columns \(\tilde{w}_i\) are such that \(\sum_{j=1}^{s} \tilde{w}_i (j) \leq -\eta\). This is immediately true for the first \(s\) columns, as \(\tilde{\Gamma}_{s}^T \tilde{W} \leq -\eta \tilde{\Gamma}_{s}^T\) by assumption. In fact, see (34), we have \(\tilde{w}_i (j) = w_i (j)\) if \(1 \leq j \leq s\), 0 otherwise.

Finally, for the last column \(\tilde{w}_{s+1}\), from (33) we have that
\[
\left( \sum_{i=1}^{s} \xi_i \right) + \zeta = \alpha_1 \left( \sum_{i=1}^{s} w_1 (i) \right) + \cdots + \alpha_s \left( \sum_{i=1}^{s} w_s (i) \right) \leq -\eta \\left( \sum_{i=1}^{s} \alpha_i \right) = -\eta.
\]

\[\text{\(\blacksquare\)}\]
B. Proof of Theorem 10

Proof: For each vertex \( x^1_k \) of \( L_{V_1} \), say \( j = 1 \), (13a) is equivalent to the contraction of \( V_1 \) in \( x^1_k \), namely to having \( V_1(x^1_k) < 0 \). Then (13b) is equivalent to the contraction of \( V_2 \) in \( x^1_k \). Since \( V_2 \) is homogeneous and the condition holds for all \( i \in [1, N] \), this is equivalent to the contraction of \( V_2 \) in \( x^1_k \) itself.

The common controller follows from \( \tilde{U} \) and therefore it is piecewise-linear. The proof immediately follows since the choice of \( j \in \{1, 2\} \) and \( k \in [1, s_j] \) have been made arbitrarily.

C. Proof of Theorem 11

Proof: The assumption that \( V_1 \) is a PCLF is equivalent to the existence of a piecewise-linear controller that follows from the control vectors \( u_1, u_2, ..., u_s \) (respectively associated with the vertices \( x_1, x_2, ..., x_s \) ), namely the columns of \( U \), which shows up in (11). According to Lemma 9, if \( \{x_1, x_2, ..., x_s\} \) are the vertices of a given facet of the polyhedron \( L_{V_1} \), together with control vectors \( \{u_1, u_2, ..., u_s\} \), then the control vector \( \tilde{u}(\alpha) := \sum_{h=1}^r \alpha_h u_h \), for \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_r) \in A := \{ \alpha \in (\mathbb{R}^\geq 0)^r \mid \sum_{h=1}^r \alpha_h = 1 \} \), is an admissible control for \( V_1 \) in the state point \( \tilde{x}(\alpha) := \sum_{h=1}^r \alpha_h x_h \).

Therefore it is sufficient to prove that for each facet of the polyhedron \( L_{V_1} \), the control \( \tilde{u}(\alpha) \), parameterized by \( \alpha \in A \), is admissible also for \( V_2 \), i.e. there exists \( \eta > 0 \) such that

\[
\tilde{x}(\alpha)^T P [(A_i + \eta I_n)\tilde{x}(\alpha) + B_i\tilde{u}(\alpha)] \leq 0 \quad \forall i \in [1, N].
\]

Then we can write

\[
\left( \sum_{h=1}^r \alpha_h x_h \right)^T P \left( (A_i + \eta I_n) \sum_{h=1}^r \alpha_h x_h \right) + B_i \left( \sum_{h=1}^r \alpha_h u_h \right) \leq 0 \quad \forall i \in [1, N] \iff \sum_{h=1}^r \alpha_h \alpha_j \left( x_j^T P [(A_i + \eta I_n)x_j + B_i u_j] \right) \leq 0 \quad \forall i \in [1, N].
\]

We get the left-hand side of the last inequality, namely \( \alpha^T S_i(\eta, \eta) \alpha \), to be non-positive for \( \alpha \in (\mathbb{R}^\geq 0)^r \). Therefore the matrices \( -S_{i,j}(\eta, \eta) \), where the subscript \( k \) indicates the \( k \)-th facet, have to be copositive. This is equivalent to the assumption made.
also $V$, satisfies the small control property. The optimization problem (22) follows from the minimal selection control

$$m(x) := \arg \min \left\{ \|v\| : \max_{i \in [1,N]} \nabla V_i(x) A_i x + \nabla V_i(x) B_i (v + \kappa_2(x)) \leq -\eta x^T x \right\},$$

that is known to be continuous [31, Section 4.2]. Hence the optimal solution of (22) can be written as $\kappa(x) := m(x) + \kappa_2(x)$, which is the sum of two continuous functions.

In the following, we prove that $\kappa_2$ is an admissible control for $V_\lambda$ in a neighborhood of the origin. This will also imply that $V_\lambda$ satisfies the small control property.

According to Lemma 20, we have the following property.

For any $\epsilon > 0$ and $\sigma \in (0,1)$ there exists $\phi > 0$ such that $\phi > \phi$ implies that $\nabla V_\lambda(x) = \nabla V_2(x) + v(x)^T$, with $\max_{x \in \mathcal{L}(V_\lambda, \sigma)} \|v(x)\| \leq \epsilon$. Therefore

$$\max_{i \in [1,N]} \nabla V_\lambda(x)(A_i x + B_i \kappa_2(x)) = \max_{i \in [1,N]} \nabla V_2(x)(A_i x + B_i \kappa_2(x)) + v(x)^T (A_i x + B_i \kappa_2(x)) \leq \nabla V_2(x)(A_i x + B_i \kappa_2(x)) + \max_{i \in [1,N]} v(x)^T (A_i x + B_i \kappa_2(x)).$$

We notice that there exists $\eta, \sigma > 0$ such that $\max_{i \in [1,N]} \nabla V_\lambda(x)(A_i x + B_i \kappa_2(x)) \leq -2\eta x^T x$ for all $x$ in the compact set $\mathcal{L}(V_\lambda, \sigma)$. Therefore we choose $\sigma$ so that $\{x \in \mathbb{R}^n \mid V_\lambda(x) \leq \sigma \} \subseteq \{x \in \mathbb{R}^n \mid V_2(x) \leq \sigma_2 \}$, namely as $\sigma := \max\{c \in [0,1] \mid \mathcal{L}(V_\lambda, c) \subseteq \mathcal{L}(V_2, \sigma_2)\}$.

We can now choose $\epsilon \geq \|v(x)\|$ such that

$$\max_{x \in \mathcal{L}(V_\lambda, \sigma)} \left\{ \max_{i \in [1,N]} v(x)^T (A_i x + B_i \kappa_2(x)) - \eta x^T x \right\} \leq 0.$$ (37)

Therefore, using (37) in (36), we get that $\kappa_2$ is an admissible control for $V_\lambda$ in a neighborhood of the origin, i.e.

$$\max_{i \in [1,N]} \nabla V_\lambda(x)(A_i x + B_i \kappa_2(x)) \leq -\eta x^T x.\$$

This means that for all $x \in \mathcal{L}(V_\lambda, \sigma)$, the constraint $v \in U(x)$ in (22) is not active and therefore $\kappa(x) = \kappa_2(x)$ is locally optimal. Moreover, we also get that the controller $\kappa$ is continuous also at the origin. \hfill \blacksquare

**Appendix E**

**Technical properties of the $R$-composition**

**Lemma 17.** $V_\lambda$ is positive definite.

**Proof:** At the origin we have $V_1(0) = V_2(0) = 0 \iff R_1(0) = R_2(0) = 1$. Therefore, from (17), $R_\lambda(0) = 1$ and hence $V_\lambda(0) = 1 - R_\lambda(0) = 0$. Conversely, $V_\lambda(x) = 0 \iff R_\lambda(x) = 1$. From (23) Proposition 1, we have $1 = R_\lambda(x) \leq \max\{R_1(x), R_2(x)\}$. Since $R_1(x) \leq 1$ and $R_2(x) \leq 1$ by construction, we have that $R_1(x) = 1$ or $R_2(x) = 1$ (or both). Say $R_1(x) = 1$. Therefore $R_1(x) = 1 \iff V_1(x) = 0 \iff x = 0$. \hfill \blacksquare

**Lemma 18.** Assume that $V_1$ and $V_2$ are differentiable respectively in $\mathcal{L}_V$ and $\mathcal{L}_V$. Then $V_\lambda$ is differentiable in $\text{int}\mathcal{L}_V$.

**Proof:** The proof immediately follows from (19) since $\phi > 0$ is fixed and functions $c_i(\phi, x)$, $i = 1, 2$, are continuous whenever $R_1(x)$ and $R_2(x)$ are not simultaneously 0, i.e. in $\text{int}\mathcal{L}_V$. \hfill \blacksquare

For ease of notation, in the following proofs, let us denote $V_1(x), V_2(x), R_1(x), R_2(x), c_1(\phi, x), c_2(\phi, x)$ without the explicit dependence on their arguments.

**Lemma 19.** $\mathcal{L}_V_\lambda = \mathcal{L}_V_1 \cap \mathcal{L}_V_2$.

**Proof:** According to (23) Lemma 1, we have $R_\lambda > 0 \iff \{R_1 > 0 \land R_2 > 0\}$; moreover, from (17), $R_\lambda = 0 \iff \{R_1 = 0 \lor R_2 = 0\}$. Now by construction $V_\lambda = 1 - R_1$, $i \in \{1,2\}$, and $V_\lambda = 1 - R_\lambda$, therefore $V_\lambda < 1 \iff \{V_1 < 1 \land V_2 < 1\}$, and $V_\lambda = 1 \iff \{V_1 = 1 \lor V_2 = 1\}$, i.e. $\mathcal{L}_V_\lambda = \mathcal{L}_V_1 \cap \mathcal{L}_V_2$. \hfill \blacksquare

**Lemma 20.** $\nabla V_\lambda$ converges to $\nabla V_2$ uniformly on compact subsets of $\text{int}\mathcal{L}_V_\lambda$, as $\phi \to \infty$. Namely, for any $\delta \in (0,1)$ we have

$$\lim_{\phi \to \infty} \max_{x \in \mathcal{L}(V_\lambda, \delta)} \|\nabla V_\lambda(x) - \nabla V_2(x)\| = 0.$$

**Proof:** First we have

$$\lim_{\phi \to \infty} \rho(\phi) = \lim_{\phi \to \infty} \frac{1}{\phi + 1 - \sqrt{\phi^2 + 1}} = \lim_{\phi \to \infty} \frac{\phi + 1 + \sqrt{\phi^2 + 1}}{2\phi} = 1. \quad (38)$$

Then

$$\lim_{\phi \to \infty} \rho c_1 = \lim_{\phi \to \infty} \phi \left(1 + \frac{-\phi R_1}{\sqrt{\phi^2 R_1^2 + R_2^2}}\right) = \frac{R_2^2}{\phi^2 R_1^2 + \phi R_2 + R_1 \sqrt{\phi^2 R_2^2 + R_2^2}} \leq \frac{1}{2\phi R_1^2} \leq \lim_{\phi \to \infty} \frac{1}{2\phi} (1 - \delta)^2 = 0. \quad (39)$$

The last inequality holds uniformly as $R_1(x) \geq 1 - \delta > 0$ whenever $x \in \mathcal{L}(V_\lambda, \delta) = \{y \in \mathbb{R}^n \mid V_\lambda(y) \leq \delta\}$. Then we can also write

$$\lim_{\phi \to \infty} \rho c_2 = \lim_{\phi \to \infty} \phi \left(1 + \frac{-R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}}\right) = \frac{\sqrt{\phi^2 R_1^2 + R_2^2} - R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}} = \frac{R_2^2}{\phi^2 R_1^2 + R_2^2} = 1. \quad (40)$$

Therefore, combining (38), (39) and (40), we get

$$\lim_{\phi \to \infty} \nabla V_\lambda(x) = \lim_{\phi \to \infty} \rho(\phi) c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x) = \nabla V_2(x)$$

uniformly on compact subsets of the kind $\mathcal{L}(V_\lambda, \delta)$. \hfill \blacksquare

**Lemma 21.** $\nabla V_\lambda$ converges to $\nabla V_1$ uniformly on compact subsets of $\text{int}\mathcal{L}_V$, as $\phi \to 0^+$. Namely, for any $\delta \in (0,1)$ we have

$$\lim_{\phi \to 0^+} \max_{x \in \mathcal{L}(V_\lambda, \delta)} \|\nabla V_\lambda(x) - \nabla V_1(x)\| = 0.$$

**Proof:** Since $\nabla V_\lambda = \rho(\phi) [c_1 \nabla V_1 + c_2 \nabla V_2]$, we have to prove that for any $\delta \in (0,1)$ we have
\[ \lim_{x \to 0^+} \rho(\phi) c_1(x) = 1 \quad \text{and} \quad \lim_{x \to 0^+} \rho(\phi) c_2(\phi, x) = 0 \quad \text{for all} \quad x \in L(V_\wedge, \beta). \]

Similarly to (38) and (39) we have that
\[
\lim_{\phi \to 0^+} \rho(\phi) c_1 = 0,
\lim_{\phi \to 0^+} \rho(\phi) c_2 = 0,
\lim_{\phi \to 0^+} \frac{\phi + 1 - \sqrt{\phi^2 + 1}}{2\phi} \cdot \frac{\phi R_2^2}{\phi^2 R_1^2 + R_2^2 + \phi R_1 \sqrt{\phi^2 R_1^2 + R_2^2}} = 1. \tag{41}
\]

The last equality holds uniformly as \( R_1(x) \geq 1 - \delta > 0 \) and \( R_2(x) \geq 1 - \delta > 0 \) (both the numerator and the denominator of (41) and also the denominator of (42) are strictly positive) whenever \( x \in L(V_\wedge, \beta) = \{ y \in \mathbb{R}^n | V_\wedge(y) \leq \beta \} \).

Then we can write
\[
\lim_{\phi \to 0^+} \rho(\phi) c_2 = 0. \tag{42}
\]

Since \( R_1(x), R_2(x) \geq 1 - \delta > 0 \), the denominator is strictly positive and hence the last equality holds uniformly. Therefore, from (41) and (42) we get \( \lim_{\phi \to 0^+} \rho(\phi) (c_1(\phi, x) + c_2(\phi, x) \nabla V_1(x)) = \nabla V_1(x) \) uniformly on compact subsets of the kind \( L(V_\wedge, \beta) \).

**Lemma 22.** Assume \( L_{V_2} \supset L_{V_1} \). Then \( \nabla V_\wedge \) converges to \( \nabla V_1 \) uniformly on \( L_{V_\wedge} \) as \( \phi \to 0^+ \), i.e.
\[
\lim_{\phi \to 0^+} \max_{x \in L_{V_\wedge}} \| \nabla V_\wedge(x) - \nabla V_1(x) \| = 0. \tag{43}
\]

**Proof:** We first notice that, as \( L_{V_2} \supset L_{V_1} \), we have \( L_{V_\wedge} = L_{V_1} \) in view of Lemma 19. Then we can use the same proof of Lemma 21 if we notice that \( R_2(x) \) is strictly positive in \( L_{V_\wedge} \) because \( L_{V_2} \supset L_{V_1} = L_{V_\wedge} \). In fact, \( R_2(x) > 0 \) implies that both the numerator and the denominator of (41), and also the denominator of (42), are strictly positive for all \( x \in L_{V_\wedge} \).

**REFERENCES**


